

Math 213 Exam 2 (Solutions)

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Problem 1

A binary string x of length 20 is chosen at random. Let X be the random variable where $X(x)$ is the number of 0s in x plus twice the number of 1s in x . Find $E(X)$ and $V(X)$.

Solution.

For $i = 1, 2, \dots, 20$ let

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th bit is 0} \\ 2, & \text{if the } i\text{-bit is 1} \end{cases}$$

Then $X = X_1 + \dots + X_{20}$. Clearly, the random variables X_1, \dots, X_{20} are independent.

Therefore $EX = \sum_{i=1}^{20} EX_i = 20EX_1$ and $VX = \sum_{i=1}^{20} VX_i = 20VX_1$. We have

$$EX_1 = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2}$$

and

$$E(X_1^2) = 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = \frac{5}{2}$$

Therefore

$$EX = \sum_{i=1}^{20} EX_i = 20EX_1 = 20 \cdot \frac{3}{2} = 30,$$

and

$$V(X) = 20VX_1 = 20(E(X_1^2) - (EX_1)^2) = 20\left(\frac{5}{2} - \frac{9}{4}\right) = 20 \cdot \frac{1}{4} = 5.$$

Problem 2

Find the general solution of the recurrence relation

$$(\dagger) \quad a_n = 4a_{n-1} - 4a_{n-2} + n.$$

Solution.

The associated homogeneous relation is $a_n = 4a_{n-1} - 4a_{n-2}$. It has characteristic equation $r^2 = 4r - 4$, that is

$$r^2 - 4r + 4 = (r - 2)^2.$$

Hence the homogeneous relation $a_n = 4a_{n-1} - 4a_{n-2}$ has the general solution $a_n^{(h)} = (c_1 + c_2n)2^n$, where $c_1, c_2 \in \mathbb{R}$.

We can look for a particular solution of the relation (\dagger) of the form $a_n = \alpha + \beta n$. Substitution this expression in (\dagger) we get

$$\begin{aligned}\alpha + \beta n &= 4(\alpha + \beta(n-1)) - 4(\alpha + \beta(n-2)) + n \\ (\beta - 1)n + \alpha - 4\beta &= 0, \quad \text{and hence} \\ \beta - 1 = 0, \alpha - 4\beta &= 0 \\ \beta = 1, \alpha = 4.\end{aligned}$$

Thus $a_n = 4 + n$ is a particular solution of (†). Therefore (†) has general solution

$$a_n = (c_1 + c_2 n)2^n + 4 + n,$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Problem 3

Find the number of onto functions from a 5-element set to a 3-element set.

Solution.

The number of onto functions from a 5-element set to a 3-element set is:

$$3^5 - C(3,1)(3-1)^5 + C(3,2)(3-2)^5 = 243 - 3 \cdot 32 + 3 \cdot 1 = 150.$$

Problem 4

Let A be the set of all nonempty strings of uppercase and lowercase English letters.

Let R be the set of all pairs (x, y) where $x, y \in A$ are strings of the same length such that for every position n the n -th characters of x and y are the same English letter, either uppercase or lowercase.

- (a) Show that R is an equivalence relation on A .
- (b) Find the equivalence class of the string *Yes*.

Solution.

(a) It is clear that for every $x \in A$ we have xRx and that R is symmetric. Let $x, y, z \in A$ be such that xRy and yRz . Since xRy and yRz , the string y has the same length as x and also y has the same length as z . Hence x and z have the same length. Moreover, for every position i the i -th letter of y is the same (with a possible difference in case) as the i -th letter of x and as the i -th letter of z . Hence the i -th characters of x and z are the same, with a possible difference in case. Thus xRz and R is transitive. Therefore R is an equivalence relation.

- (b) The equivalence class of *Yes* is:

$$\{yes, Yes, yEs, yeS, YEs, YeS, yES, YES\}.$$

Problem 5

Find the conditional probability that exactly four heads appear when a fair coin is tossed 5 times, given that the first flip came up heads.

Let E be the event that exactly four heads appear and let F be the event that the first flip is a head.

Then $E \cap F$ is the event that the first flip is a head and that among the remaining 4 flips exactly three are heads.

Therefore

$$P(F) = \frac{1 \cdot 2^4}{2^5}$$

and

$$P(E \cap F) = \frac{1 \cdot C(4, 3)}{2^5}.$$

Hence

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{C(4, 3)}{2^4} = \frac{4}{16} = \frac{1}{4}.$$