

**Math 213 Exam 2 9Solutions)**

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**Problem 1**[20 points]

For each of the following statements indicate whether it is true or false. You DO NOT have to explain your answers.

- (1) For any random variables  $X$  and  $Y$  we have  $V(X + Y) = VX + VY$ .
- (2) For  $S = \{-5, 3, 1, 5\}$ , ordered as listed, the permutation  $3, 1, 5, -5$  is a derangement.
- (3) Let  $A$  be the set of all bit-strings of length 10. Let  $R$  be a relation on  $A$  consisting of all pairs  $(x, y) \in A \times A$  such that the strings  $x$  and  $y$  agree in their first two bits.

Then  $R$  is an equivalence relation with exactly two equivalence classes.

- (4) Consider a recurrence relation  $a_n = 3a_{n-1} + n$ .  
Then for every  $c, \alpha, \beta \in \mathbb{R}$ ,  $a_n = c3^n + \alpha + \beta n$  is a solution of the above recurrence relation.

**Answers:**

(1) False. (The statement would be true if we knew that  $X$  and  $Y$  are independent random variables).

(2) True.

(3) False. (Although  $R$  is an equivalence relation, it has four distinct equivalence classes, corresponding to 4 distinct binary strings of length two that can occur as initial segments of length two of strings from  $A$ ).

(4) False. (One can find a particular solution of this recurrence relation of the form  $a_n = \alpha + \beta n$ , but the values of  $\alpha$  and  $\beta$  that work are not arbitrary. In this case these specific values are  $\alpha = -\frac{3}{4}$ ,  $\beta = -\frac{1}{2}$  and the general solution of the original recurrence relation is  $a_n = c3^n - \frac{3}{4} - \frac{1}{2}$ , where  $c \in \mathbb{R}$  is arbitrary.).

**Problem 2**[20 points]

Let  $n \geq 1$  be an integer. A fair die is rolled  $n$  times, independently. Let  $Z = 3X + 2Y$  where  $X$  is the number of tosses that came 1 up and where  $Y$  is the number of tosses that came 6 up.

Find  $EZ$  and  $VZ$ .

Give all the details of your work.

**Solution.**

For  $i = 1, \dots, n$  let

$$Z_i = \begin{cases} 3, & \text{if the } i\text{-th toss is 1,} \\ 2, & \text{if the } i\text{-th toss is 6, 0, otherwise.} \end{cases}$$

Then  $Z = Z_1 + \dots + Z_n$  and the random variables  $Z_1, \dots, Z_n$  are independent.

For each  $i = 1, \dots, n$  we have

$$EZ_i = 3\frac{1}{6} + 2\frac{1}{6} + 0\frac{4}{6} = \frac{5}{6}$$

and

$$E(Z_i^2) = 3^2\frac{1}{6} + 2^2\frac{1}{6} + 0^2\frac{4}{6} = \frac{13}{6}.$$

Hence  $VZ_i = E(Z_i^2) - (EZ_i)^2 = \frac{13}{6} - \frac{25}{36} = \frac{53}{36}$ .

Since  $Z = Z_1 + \dots + Z_n$ , it follows that

$$EZ = EZ_1 + \dots + EZ_n = \frac{5n}{6}$$

and, since  $Z_1, \dots, Z_n$  are independent,

$$VZ = VZ_1 + \dots + VZ_n = \frac{53n}{36}.$$

**Problem 3**[20 points]

Find the general solution of the recurrence relation

$$(\dagger) \quad a_n = 2a_{n-1} - a_{n-2} + 4.$$

Give all the details of your work.

**Solution.**

The associated homogeneous relation is  $a_n = 2a_{n-1} - a_{n-2}$ . Its characteristic equation is  $r^2 = 2r - 1$  which factors as  $(r - 1)^2 = 0$ . Hence the homogeneous problem  $a_n = 2a_{n-1} - a_{n-2}$  has the general solution

$$a_n^{(h)} = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

where  $\alpha, \beta \in \mathbb{R}$  are arbitrary constants. Since  $4 = 4 \cdot 1^n$ , we should look for a particular solution of  $(\dagger)$  of the form

$$a_n^{(p)} = cn^2,$$

where  $c \in \mathbb{R}$ . To find the specific value of  $c$ , we substitute  $a_n^{(p)} = cn^2$  into  $(\dagger)$ :

$$\begin{aligned} cn^2 &= 2c(n-1)^2 - c(n-2)^2 + 4 \\ cn^2 &= 2cn^2 - 4cn + 2c - cn^2 + 4cn - 4c + 4 \\ 2c &= 4, \quad c = 2. \end{aligned}$$

Thus  $a_n^{(p)} = 2n^2$  is a particular solution of  $(\dagger)$ . Therefore the general solution of  $(\dagger)$  is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha + \beta n + 2n^2,$$

where  $\alpha, \beta \in \mathbb{R}$  are arbitrary constants.

**Problem 4**[20 points]

What is the number of ways to distribute five different toys to three children, so that every child gets at least one toy?

Give all the details of your work.

**Solution.**

We have to count the number of onto functions from the set of five toys to the set of three children. This number is:

$$3^5 - 2^5 C(3, 2) + 1^5 C(3, 1) = 243 - 32 \cdot 3 + 3 = 150.$$

**Problem 5**[20 points]

Let  $A = \{1, 2, 3\}$ . Let  $R = \{(1, 2), (2, 2), (2, 3), (3, 1)\}$  and let  $S = \{(2, 3)\}$  be relations on  $A$ .

(a) Compute the relations  $S \circ R$  and  $R \circ S$ .

(b) Is  $R$  transitive? Is  $S$  transitive?

Give all the details of your work.

**Solution.**

(a) Recall that by definition of composition of relations,

$$S \circ R = \{(a, b) \in A \times A : \text{there exists } c \in A \text{ such that } (a, c) \in R, (c, b) \in S\}.$$

We have  $(1, 2) \in R, (2, 3) \in S$  and hence  $(1, 3) \in S \circ R$ . Also,  $(2, 2) \in R, (2, 3) \in S$  and hence  $(2, 3) \in S \circ R$ . There are no other  $a, b, c$  such that  $(a, c) \in R, (c, b) \in S$ . Therefore  $S \circ R = \{(1, 3), (2, 3)\}$ .

Similarly,  $(2, 3) \in S, (3, 1) \in R$  and hence  $(2, 1) \in R \circ S$ . There are no other  $a, b, c$  such that  $(a, b) \in S, (b, c) \in R$ . Therefore  $R \circ S = \{(2, 1)\}$ .

(b) We have  $(1, 2), (2, 3) \in R$  but  $(1, 3) \notin R$ . Therefore  $R$  is not transitive.

The relation  $S$  is transitive since there are no  $a, b, c \in A$  such that  $(a, b), (b, c) \in S$  and hence there is nothing to check and the condition of being transitive holds by default.