Math 231 Midterm Exam 1 (Solutions)
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Problem 1.[20 points]

For each of the following statements indicate whether it is true or false. You DO NOT need to provide explanations for your answers in this problem.

(1) For the rational function $\frac{ax^3+b}{x^2(x^2+1)^2}$ the form of its partial fractions decomposition is

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^2+1}+\frac{D}{(x^2+1)^2},$$

where A, B, C, D are some constants.

(2) For integrals involving $\sqrt{x^2-a^2}$ the corresponding trigonometric substitution is $x = a\sin\theta$.

(3) The integration by parts formula is a consequence of the Product Rule for differentiation.

(4) For the trigonometric integrals of the form $\int \tan^m x \sec^n x\, dx$ where m is odd the appropriate substitution to use is $u = \sec x$.

(5) If $f(x) = x^3$ and $a \le b$, then for every even $n \ge 1$ we have $\int_a^b f(x)\, dx = S_n$ where S_n is the approximation for $\int_a^b f(x)\, dx$ using Simpson's Rule.

Answers:

(1) FALSE.

The form of the partial fractions decomposition here is:

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C_1x+C_2}{x^2+1}+\frac{D_1x+D_2}{(x^2+1)^2},$$

where A, B, C_1, C_2, D_1, D_2 are some constants.

(2) FALSE.

The corresponding trigonometric substitution here is $x = a\sec\theta$.

(3) TRUE.

(4) TRUE.

(5) TRUE.

The error estimate for Simpson's Rule is $|E_S| \le \frac{K(b-a)^5}{180n^4}$ where K is such that the fourth derivative $|f^{(4)}(x)| \le K$ on $[a, b]$. For $f(x) = x^3$ we have $f' = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(4)} = 0$ on $[a, b]$. Hence we can use $K = 0$ which implies that $|E_S| \le 0$, so that $E_S = 0$. Thus $\int_a^b f(x)\, dx - S_n = E_S = 0$ and $S_n = \int_a^b f(x)\, dx$.

Problem 2.[20 points]

Compute the following integral and give all the details of your work:

$$\int \frac{1}{\sqrt{13-4x+x^2}}\, dx$$

Solution.
We have

$$\int \frac{1}{\sqrt{13-4x+x^2}}\,dx = \int \frac{1}{\sqrt{9+4-4x+x^2}}\,dx =$$
$$\int \frac{1}{\sqrt{9+(x-2)^2}}\,dx = \int \frac{1}{\sqrt{3^2+y^2}}\,dy$$

where $y = x-2$, $dy = dx$.

We then use a trigonometric substitution $y = 3\tan\theta$, with $\sqrt{9+y^2} = 3\sec\theta$. Then $y = 3\tan\theta$, $dy = 3\sec^2\theta\,d\theta$ and

$$\int \frac{1}{\sqrt{3^2+y^2}}\,dy = \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta} =$$
$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C =$$
$$\ln\left|\frac{\sqrt{9+(x-2)^2}}{3} + \frac{x-2}{3}\right| + C = \ln\left|\frac{\sqrt{13-4x+x^2}}{3} + \frac{x-2}{3}\right| + C$$

Problem 3.[20 points]
Compute the following integral:

$$\int \frac{x^2+1}{(x-1)^2(x+1)}\,dx$$

Show the details of your work.

Solution. We first find the partial fractions decomposition of the integrand:

$$\frac{x^2+1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{x+1} =$$
$$\frac{A(x-1)(x+1) + B(x+1) + D(x-1)^2}{(x-1)^2(x+1)}$$

We need to find constants A, B, D such that $A(x-1)(x+1) + B(x+1) + D(x-1)^2 = x^2+1$. Substituting the values $x = 1, -1, 0$ in this equation we get:

$$A\cdot 0 + 2B + D\cdot 0 = 1^2 + 1 = 2, \quad \Longrightarrow \quad B = 1$$
$$A\cdot 0 + B\cdot 0 + 4D = (-1)^2 + 1 = 2, \quad \Longrightarrow \quad D = \frac{1}{2}$$
$$-A + B + D = 1, \quad \Longrightarrow \quad A = B + D - 1 = \frac{1}{2}.$$

Hence

$$\int \frac{x^2+1}{(x-1)^2(x+1)}\,dx = \int \frac{1}{2}\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{2}\frac{1}{x+1}\,dx =$$
$$\frac{1}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + C.$$

Problem 4.[20 points]

Let $f(x)$ be a function such that $f'(x) = \ln(x^2+2)$. Let T_5 be the approximation of $\int_1^3 f(x)\,dx$ obtained using the Trapezoidal Rule with $n=5$. Find an estimate for the absolute value $|E_T|$ of the error in this approximation. Give a careful justification of your answer and show all the details of your work.

Solution.

We have $f''(x) = \frac{2x}{x^2+2}$. Since $1 \le x \le 3$ and $2x \le 2\cdot 3 = 6$, $x^2+2 \ge 1^2+2 = 3$, it follows that on the interval $[1,3]$

$$|f''(x)| = |\frac{2x}{x^2+2}| = \frac{2x}{x^2+2} \le \frac{6}{x^2+2} \le \frac{6}{1^2+2} = 2.$$

We know that for the Trapezoidal Rule the error in approximating $\int_a^b f(x)\,dx$ can be estimated as $|E_T| \le \frac{K(b-a)^3}{12n^2}$, where $|f'(x)| \le K$ on $[a,b]$. In this case we can use $K=2$, $n=5$, $a=1, b=3$, which yields:

$$|E_T| \le \frac{2(3-1)^3}{12\cdot 5^2} = \frac{16}{300} \approx 0.053333$$

Note. A more careful argument yields a smaller value of K such that $|f''(x)| \le K$ on $[1,3]$. Indeed, for $g(x) = f''(x) = \frac{2x}{x^2+2}$ we have $g'(x) = \frac{2(x^2+2)-2x\cdot 2x}{(x^2+2)^2} = \frac{4-2x^2}{(x^2+2)^2}$. Thus $g'(x)=0$ gives $4-2x^2=0$ and hence $x=\sqrt{2}$ (recall that $1 \le x \le 3$). Therefore the maximal value of $g(x)$ on $[1,3]$ is

$$\max\{g(1), g(\sqrt{2}), g(3)\} = \max\{\frac{2}{3}, \frac{\sqrt{2}}{2}, \frac{6}{11}\} = \frac{\sqrt{2}}{2} \approx 0.7071$$

Thus we can use $K = \frac{\sqrt{2}}{2}$ which yields a better error estimate:

$$|E_T| \le \frac{\frac{\sqrt{2}}{2}(3-1)^3}{12\cdot 5^2} \approx 0.018856$$

Problem 5.[20 points]

Compute the integral

$$\int \ln(2x+1)\,dx$$

Show all the details of your work.

Solution.

We use integration by parts with $u = \ln(2x+1)$, $dv = dx, v = x$.

Then

$$\int \ln(2x+1)\,dx = x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{2x+1-1}{2x+1}\,dx =$$

$$x\ln(2x+1) - \int 1 - \frac{1}{2x+1}\,dx = x\ln(2x+1) - x + \frac{1}{2}\ln(2x+1) + C.$$