Math 231 Midterm Exam 3 (Solutions).

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Problem 1.[20 points]

For each of the following statements indicate whether it is true or false. You DO NOT need to provide explanations for your answers in this problem.

(1) For every real number $x \neq 1$ we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

(2) Suppose that a power series $\sum_{n=0}^{\infty} a_n(x-2)^n$ has radius of convergence $R > 0$. Then for the function $f(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ the derivatives $f^{(n)}(2)$ exist for all $n \geq 0$.

(3) For a curve given by parametric equations $x = f(t), y = g(t)$, $0 \leq t \leq 1$, the length of this curve is equal to

$$\int_0^1 \sqrt{f''(t) + g''(t)}\, dt$$

(4) If the power series $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R = 2$ then the series $\sum_{n=1}^{\infty}(-1)^n a_n \pi^n$ diverges.

(5) We have $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+1)!6^n} = \frac{1}{2}$.

Answers.

(1) False. This equality only holds when $|x| < 1$.

(2) True. This follows, for example, from Theorem 2 in Ch. 11.9

(3) False. The length of the curve is equal to $\int_0^1 \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$.

(4) True. Since $|-\pi| \approx 3.1415926 > 2$ and the radius of convergence is $R = 2$, the power series diverges for $x = -\pi$.

(5) False. The series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+1)!6^n} = 1 - \frac{\pi}{6\cdot 3!} + \frac{\pi^2}{6^2\cdot 5!} - \dots$ is an alternating series satisfying the conditions of the Alternating Series Test. Therefore if $S = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+1)!6^n}$ is the sum of this series, then $S \approx 1 - \frac{\pi}{6\cdot 3!} = 0.91273$ with error less than $\frac{\pi^2}{6^2\cdot 5!} = 0.00228$. Therefore $S \geq 0.91273 - 0.00228 = 0.91045$ and hence $S \neq \frac{1}{2}$.

Problem 2.[20 points]

Let $f(x) = \sin(x^2)$. For every $n \geq 0$ find the value of the derivative $f^{(n)}(0)$. Give all the details of your work.

Solution

We have

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

for every $x \in \mathbb{R}$ and hence

$$(*) \qquad f(x) = \sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^2)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}$$

for every $x \in \mathbb{R}$. This series must coincide with the Maclaurin series of $f(x)$, that is

$$(**) \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

By equating the coefficients at x^n in $(*)$ and $(**)$ for every $n \geq 0$ we conclude that

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n \neq 4k+2 \\ \frac{(-1)^k (4k+2)!}{(2k+1)!}, & \text{if } n = 4k+2 \end{cases} .$$

Problem 3.[20 points]

Find the radius of convergence and the interval of convergence for the following series:

$$(\dagger) \qquad \sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}.$$

Give all the details of your work.

Solution.

Using the Ratio Test for the series $\sum_{n=0}^{\infty} a_n$ with $a_n = (-1)^n \frac{(x-3)^n}{2n+1}$ we get

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{2(n+1)+1} \frac{2n+1}{(x-3)^n} \right| =$$

$$= \lim_{n\to\infty} |x-3| \frac{2n+3}{2n+1} = |x-3|.$$

Hence, by the Ratio Test, the series $(\dagger)$ converges when $|x-3| < 1$ and diverges when $|x-3| > 1$. Therefore the radius of convergence is $R = 1$.

The endpoints of the interval $|x-3| \leq 1$ are $x = 2$ and $x = 4$. For $x = 2$ the series $(\dagger)$ takes the form:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2-3)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

This series diverges, for instance by the Integral Test.

For $x = 4$ the series (†) takes the form:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(4-3)^n}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

This series converges by the Alternating Series Test.

Thus the interval of convergence for (†) is $2 < x \leq 4$, that is $(2, 4]$.

Problem 4.[20 points] Let $f(x) = \cos(x^2+1)$. Use Taylor's Inequality to estimate from above $|R_1(x)|$ for the Maclaurin series of the function $f(x)$ on the interval $|x| \leq 1$. Give all the details of your work.

Solution.

We have

$f'(x) = -2x\sin(x^2+1)$ and $f''(x) = -2\sin(x^2+1) - 4x^2\cos(x^2+1)$. Therefore on the interval $|x| \leq 1$ we have

$$|f''(x)| = |-2\sin(x^2+1)-4x^2\cos(x^2+1)| \leq 2|\sin(x^2+1)|+4x^2|\cos(x^2+1)| \leq 6,$$

so that we can take $M = 6$. Hence by Taylor's Inequality on the interval $|x| \leq 1$ we have

$$|R_1(x)| \leq \frac{M}{2!}|x-0|^2 = \frac{6}{2}x^2 = 3x^2.$$

Problem 5.[20 points]

Consider the parametric curve given by equation $x = e^t$, $y = t^3 + t + 1$, where $-1 \leq t \leq 1$. Find the value $\frac{d^2y}{dx^2}|_{t=0}$.

Give all the details of your work.

Solution.

We have $\frac{dy}{dt} = 3t^2 + 1$ and $\frac{dx}{dt} = e^t$. Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2+1}{e^t}.$$

Then

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{6t \cdot e^t - (3t^2+1)\cdot e^t}{e^{2t}} = \frac{6t-3t^2-1}{e^t}.$$

Therefore

$$\frac{d^2y}{dx^2}|_{t=0} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}|_{t=0} = \frac{6t-3t^2-1}{e^{2t}}|_{t=0} = -1.$$