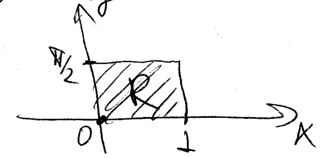
5.5 no 5 $\boxed{\text{H/wk 10}}$ (

a) $$\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\,dx\,dy =$$
$$= (b-a)\,\text{area}(R),$$ where R - region bounded by C

b) $$\oint_C e^x \sin y\,dx + e^x \cos y\,dy$$

C: rectangle with vertices $(0,0)$, $(1,0)$, $(1, \frac{1}{2}\pi)$, $(0, \frac{1}{2}\pi)$

$$= \iint_R (e^x \cos y - e^x \cos y)\,dx\,dy = 0$$



c) $$\oint_{x^2+y^2=1} (2x^3-y^3)\,dx + (x^3+y^3)\,dy = \iint_{x^2+y^2\le 1} (3x^2+3y^2)\,dx\,dy =$$

$$= 3\iint_{x^2+y^2\le 1} (x^2+y^2)\,dx\,dy \underset{\substack{x=r\cos\theta \\ y=r\sin\theta}}{=} 3 \iint_{\substack{0\le r\le 1 \\ 0\le\theta\le 2\pi}} r^2 \cdot r\,dr\,d\theta =$$

$$= 3\int_0^{2\pi}\left(\int_0^1 r^3\,dr\right)d\theta = 3\int_0^{2\pi} \frac{1}{4}\,d\theta = 3\pi/2$$

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d) $\oint_C \vec{u}_T \, ds$ $\qquad \vec{u} = \text{grad}(x^2 y) = [2xy, x^2] = 2xy\vec{i} + x^2\vec{j}$

$C: x^2+y^2=1$

$$= \iint_{x^2+y^2 \le 1} \text{curl}_z \vec{u} \, dxdy$$

$$\text{curl}\, \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 & 0 \end{vmatrix}$$

$$\text{curl}_z \vec{u} = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy) =$$

$$= \iint_{x^2+y^2} 0 \, dxdy$$

$$= 0$$

e) $\oint_C v_n \, ds$ $\qquad \vec{v} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$C: x^2+y^2=1$

$$= \iint_{x^2+y^2 \le 1} \text{div}\, \vec{v} \, dxdy = \iint_{x^2+y^2 \le 1} (2x-2x) \, dxdy = 0$$

f) $\oint_C \frac{\partial}{\partial n}[(x-2)^2+y^2] \, ds$

$C: x^2+y^2=1$

$\vec{n}$ – outer normal

$\vec{v} := \text{grad}[(x-2)^2+y^2] =$

$= (2x-4)\vec{i} + 2y\vec{j}$

$$= \oint_C \vec{v} \cdot \vec{n} \, ds$$

$$= \iint_{x^2+y^2 \le 1} \text{div}\, \vec{v} \, dxdy = \iint_{x^2+y^2 \le 1} (2+2) \, dxdy = 4\pi$$

g) $\oint\limits_{x^2+y^2=1} \frac{\partial}{\partial n} \log \frac{1}{[(x-2)^2+y^2]} ds =$ $\boxed{C: x^2+y^2=1}$

$= \oint\limits_C \frac{\partial}{\partial n}\left[-\log[(x-2)^2+y^2]\right] ds =$

$= - \oint\limits_C \frac{\partial}{\partial n} \log[(x-2)^2+y^2] ds =$

$= \oint\limits_C \vec{v}\cdot\vec{n}\, ds$ where $\vec{v} = \text{grad} \log[(x-2)^2+y^2]$

$= \frac{2x-4}{x^2-4x+4+y^2}\vec{i} + \frac{2y}{x^2-4x+4+y^2}\vec{j}$

$\parallel$

$\iint\limits_{x^2+y^2\le 1} \text{div}\vec{v}\, dx dy$ $\quad \text{div}\vec{v} = 0$

$\parallel$

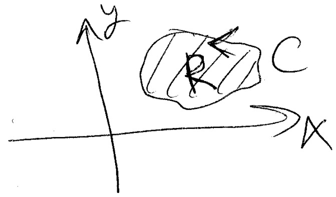
$\iint\limits_{x^2+y^2\le 1} 0\, dx dy = 0$

h) $\oint\limits_C f(x)dx + g(y)dy = \iint\limits_R \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right] dx dy =$

$= \iint\limits_R 0\, dx dy = 0$

5.5 no 6

$\vec{r} = x\vec{i} + y\vec{j}$



$\vec{n}$ – outer normal to C

$$\frac{1}{2}\oint_C r_n\, ds = \frac{1}{2}\iint_R \operatorname{div}\vec{r}\, dx\,dy = \frac{1}{2}\iint_R (1+1)\, dx\,dy =$$

$$= 2\cdot\frac{1}{2}\iint_R dx\,dy = \text{area}(R)$$

5.7 no 1

a) $dF = 2xy\,dx + x^2 dy \Rightarrow F(x,y) = x^2 y$

$$\int_{C\,(0,0)}^{(1,1)} 2xy\,dx + x^2 dy = F(1,1) - F(0,0) =$$

$$= 1$$

b) $dF = ye^{xy}dx + xe^{xy}dy \Rightarrow F(x,y) = e^{xy}$

$$\int_{C\,(0,0)}^{(\pi,0)} ye^{xy}dx + xe^{xy}dy = F(\pi,0) - F(0,0) =$$

$$= e^0 - e^0 = 0$$

c) $dF = \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} \Rightarrow F(x,y) = -(x^2+y^2)^{-1/2}$

$= -\frac{1}{\sqrt{x^2+y^2}}$

$\Rightarrow \int_{C\,(1,0)}^{(e^{2\pi},0)} \frac{x\,dx+y\,dy}{(x^2+y^2)^{3/2}} = F(e^{2\pi},0) - F(1,0) =$

$= -\frac{1}{e^{2\pi}} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{e^{2\pi}} =$

$= 1 - e^{-2\pi}$

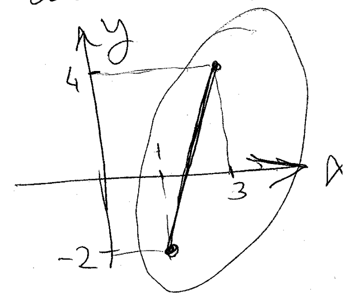
$C: \begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases} \Rightarrow$ for any t $x^2+y^2 > 0$

5.7 no 2

a) $\int_{(1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2}$ along C: line $y = 3x-5$

$\frac{y\,dx}{x^2} - \frac{x\,dy}{x^2} = \frac{y}{x}dx - \frac{1}{x}dy = dF,\ F = \left(-\frac{y}{x}\right)$

$\Rightarrow \int \frac{y\,dx - x\,dy}{x^2}$ is independent of path in any region where $x \neq 0$



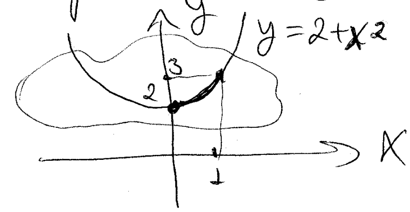
$\int_{C\,(1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2} = F(3,4) - F(1,-2) =$

$= -\frac{4}{3} - \frac{2}{1} = -\frac{10}{3}$

b) $\displaystyle\int_{(0,2)}^{(1,3)} \frac{3x^2}{y}dx - \frac{x^3}{y^2}dy$ on parabola $y=2+x^2$

$F(x,y) = \frac{x^3}{y} \Rightarrow dF = \frac{3x^2}{y}dx - \frac{x^3}{y^2}dy \Rightarrow$

integral independent of path in any domain where $y \neq 0$



Hence $\displaystyle\int_{(0,2)}^{(1,3)} \frac{3x^2}{y}dx - \frac{x^3}{y^2}dy = F(1,3) - F(0,2) =$

$= \frac{1}{3} - \frac{0}{2} = 1/3$

c) $\displaystyle\int_{(1,0)}^{(-1,0)} (2xy-1)dx + (x^2+6y)dy$ curve $y=\sqrt{1-x^2}$, $-1 \le x \le 1$

For $F(x,y) = x^2y - x + 3y^2$

$dF = (2xy-1)dx + (x^2+6y)dy \Rightarrow$

$\int (2xy-1)dx + (x^2+6y)dy$ is independent of path in xy-plane

$\displaystyle\int_{(1,0)}^{(-1,0)} (2xy-1)dx + (x^2+6y)dy = F(-1,0) - F(1,0) =$

$= 2$

d) $\displaystyle\int_{(0,0)}^{(\pi/4,\,\pi/4)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$

curve $y = 16x^3/\pi^2$

$F(x,y) = \tan x \tan y \Rightarrow$

$dF = \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$

$\Rightarrow \int \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$ is independe of path in any region where $\tan x$, $\tan y$ are defined

e.g. in the region $-\pi/2 < x < \pi/2$

$\pi/2 < y < \pi/2$

$$\Rightarrow \int_{(0,0)}^{(\pi/4,\,\pi/4)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy =$$

$$= F(\pi/4, \pi/4) - F(0,0) =$$

$$= \tan \pi/4 \cdot \tan \pi/4 - \tan 0 \tan 0 = 1.$$