(1)

H/wk 12

#5. (a) $\iint_S x\,dydz + y\,dzdx + z\,dxdy$, $x+y+z=1$, $x=x$ $y=y$ $z=1-x-y$

$= \iint_{R_{xy}} -x(-1) - y(-1) + (1-y-x)\,dydx$

$= \int_0^1 \int_0^{1-x} (x+y+1-y-x)\,dydx$

$= \int_0^1 \int_0^{1-x} dydx$

$= \int_0^1 y\Big|_0^{1-x} = \int_0^1 1-x\,dx = x - \frac{x}{2}\Big|_0^1 = \frac{1}{2}$

(b) $\iint_S dydz + dzdx + dxdy$, $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$

$\frac{\partial(y,z)}{\partial(\phi,\theta)} = \sin^2\phi\cos\theta$

$\frac{\partial(z,x)}{\partial(\phi,\theta)} = \sin^2\phi\sin\theta$

$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \sin\phi\cos\phi$

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} [\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi]\,d\phi\,d\theta$

$= \int_0^{2\pi} [\frac{\pi}{4}\cos\theta + \frac{\pi}{4}\sin\theta + \frac{1}{2}]\,d\theta = \pi$.

(c) $\iint_S x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma = \iint_R x\,dydz + y\,dz\,dx + z\,dx\,dy$

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \cos^2\phi\sin\phi\,)\,d($

$= \int_0^{2\pi} (\frac{2}{3}\cos^2\theta + \frac{2}{3}\sin^2\theta + \frac{1}{3})\,d\theta$

$= \int_0^{2\pi} \frac{2}{3}\cdot\frac{1+\cos 2\theta}{2} + \frac{2}{3}\,\frac{1-\cos 2\theta}{2} + \frac{1}{3}$

$= 2\pi$.

(d) $\iint_S x^2 z \, d\sigma$

$x = \cos\theta$, $y = \sin\theta$, $z = z$ $\Rightarrow$ $E = 1$, $G = 1$, $F = 0$, $\sqrt{EG - F^2} = 1$

$$= \int_0^1 \int_0^{2\pi} \cos^2\theta \, z \, d\theta \, dz = \int_0^1 \pi z \, dz = \frac{\pi}{2}$$

#6 (a) $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$$\begin{cases} x = u+v \\ y = u-v \\ z = 1-2u \end{cases}, \quad \frac{\partial(y,z)}{\partial(u,v)} = -2, \quad \frac{\partial(z,x)}{\partial(u,v)} = -2, \quad \frac{\partial(x,y)}{\partial(u,v)} = -2$$

$$= -\int_0^{\frac{1}{2}} \int_v^{1-v} \left[(u+v)(-2) + (u-v)(-2) + (1-2u)(-2)\right] du\,dv$$

$$= \int_0^{\frac{1}{2}} \int_v^{1-v} 2 \, du\,dv = \int_0^{\frac{1}{2}} 2u \Big|_v^{1-v} dv = \int_0^{\frac{1}{2}} 2 - 4v = 2\left[v - v^2\right]\Big|_0^{\frac{1}{2}} = \frac{1}{2}$$

(b) $\iint_S dy\,dz + dz\,dx + dx\,dy$,

$x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \sin u \cos u, \quad \frac{\partial(y,z)}{\partial(u,v)} = \sin^2 u \cos v, \quad \frac{\partial(z,x)}{\partial(u,v)} = \sin^2 u \sin v$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\sin^2 u \cos v + \sin^2 u \sin v + \sin u \cos u\right] du\,dv$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^2 u \,(\cos v + \sin v) + \frac{1}{2}\sin 2u \, dv\,du$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 u \left[\sin v - \cos v\right]\Big|_0^{2\pi} + \frac{v}{2} \sin 2u \Big|_0^{2\pi} du$$

$$= \int_0^{\frac{\pi}{2}} \pi \sin 2u \, du = -\frac{\pi}{2} \cos 2u \Big|_0^{\frac{\pi}{2}} = \pi$$

(c) $\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\, d\sigma$

$x = \sin u \cos v$
$y = \sin u \sin v$
$z = \cos u$

$\frac{\partial(y,z)}{\partial(u,v)} = \sin^2 u \cos v$

$\frac{\partial(z,x)}{\partial(u,v)} = \sin^2 u \sin v$

$\frac{\partial(x,y)}{\partial(u,v)} = \cos u \sin u$.

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (\sin u \cos v)(\sin^2 u \cos v) + (\sin u \sin v)(\sin^2 u \sin v) + (\cos u)(\cos u \sin u)\; dv du$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u \; dv du$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^3 u\,(\cos^2 v + \sin^2 v) + \cos^2 u \sin u \; dv\, du.$$

$$= \int_0^{\frac{\pi}{2}} v \sin^3 u + v \cos^2 u \sin u \Big|_0^{2\pi} du$$

$$= 2\pi \int_0^{\pi/2} \sin^3 u + \cos^2 u \sin u \, du$$

$$= 2\pi \int_0^{\pi/2} \sin u \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2u\right) + \sin u \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2u\right) du$$

$$= 2\pi \int_0^{\pi/2} \sin u \, du = 2\pi.$$

(d) $\iint x^2 z \, d\sigma$ $\quad E = 1 \quad F = 0 \quad G = 1$

$$\int_0^1 \int_0^{2\pi} \cos^2 u \cdot v \, du\, dv = \pi \int_0^1 v\, dv = \frac{\pi}{2}$$

\# 9 (a) $\iint_S \omega \cdot n \, d\sigma = \iint_S xy^2z \, dy dz - 2x^3 dz dx + yz^2 dx dy$

$$= \iint_{x^2+y^2\le 1} \left[-xy^2(1-x^2-y^2)(-2x) + 2x^3(-2y) + y(1-x^2-y^2)^2 \right] dx dy$$

$$= \iint_{x^2+y^2\le 1} \left[2x^2y^2(1-x^2-y^2) - 4x^3y + y(1-x^2-y^2) \right] dx dy, \quad x = r\cos\theta \quad y = r\sin\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^2\cos^2\theta \, r^2\sin^2\theta(1-r^2\cos^2\theta - r^2\sin^2\theta) - 4r^3\cos^3\theta \, r\sin\theta + r\sin\theta(1-r^2\cos^2\theta - r^2\sin^2\theta)$$

$$= \int_0^{2\pi}\int_0^1 \left[2r^5\cos^2\theta\sin^2\theta - 2r^7\cos^2\theta\sin\theta - 4r^4\cos^3\theta\sin\theta + r^2\sin\theta - r^4\sin\theta \right] dr d\theta.$$

$$= \int_0^{2\pi} \left[\tfrac{1}{3}\cos^2\theta\sin^2\theta - \tfrac{1}{4}\cos^2\theta\sin\theta - \tfrac{4}{5}\cos^3\theta\sin\theta + \tfrac{1}{3}\sin\theta - \tfrac{1}{5}\sin\theta \right] d\theta = \frac{\pi}{48}$$

(b) $\iint_S (\omega \cdot n) d\sigma$, $\quad x = e^u\cos v$, $y = e^u\sin v$ $\quad z = \cos v\sin v$

$$\frac{\partial(y,z)}{\partial(u,v)} = e^u\sin v(\cos 2v)$$

$$\frac{\partial(z,x)}{\partial(u,v)} = -e^u\tfrac{1}{2}(\cos v + \cos 3v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = e^{2u}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \tfrac{1}{2}e^u(\sin 3v - \sin v) - e^u(\cos 3v + \cos v) - 3e^{2u} \, du dv$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2}(\sin 3v - \sin v)e^u - (\cos 3v + \cos v)e^u - \tfrac{3}{2}e^{2u}\Big|_0^1 dv$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2}(\sin 3v - \sin v)(e-1) - (\cos 3v + \cos v)(e-1) - \tfrac{3}{2}(e^2-1) dv$$

$$= \tfrac{1}{2}(e-1)\left[-\tfrac{1}{3}\cos 3v + \cos v\right] - (e-1)\left[\tfrac{1}{3}\sin v + \sin v\right] - \tfrac{3}{2}(e^2-1)v\Big|_0^{\pi/2}$$

$$= -(e-1) - \frac{3\pi}{4}(e^2-1)$$

(c) $\iint_S \frac{\partial w}{\partial n}\, d\sigma = \iint_S 2xy^2z\,dydz + 2x^2yz\,dzdx + x^2y^2\,dxdy$

$= \iint_R -2x^2y(1-x^2-y^2)(-2x) - 2x^2y(1-x^2-y^2)(-2y) + x^2y^2$

$= \iint_{x^2+y^2\le 1} 4x^2y^2(1-x^2-y^2) + 4x^2y^2(1-x^2-y^2) + x^2y^2$

$= \iint_{x^2+y^2\le 1} 8x^2y^2(1-x^2-y^2) + x^2y^2$

$= \int_0^{2\pi}\int_0^1 [8r^2\cos^2\theta\, r^2\sin^2\theta(1-r^2) + r^2\cos^2\theta\, r^2\sin^2\theta]\, r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{\sin^2\theta\cos^2\theta}{2}\, d\theta = \frac{\pi}{8}$

(d) $\iint_S \frac{\partial w}{\partial n}\, d\sigma = \iint_S 2x\,dydz - 2y\,dzdx + 2z\,dxdy$

$= \int_0^{\frac{\pi}{2}}\int_0^1 2e^u\cos v\, e^u\sin v\cos 2v - 2e^u\sin v(-e^u\cos v\cos 2v) + 2\cos v\sin v\cdot e^{2u})\,du\,dv$

$= \int_0^{2\pi}\int_0^1 4e^{2u}\cos v\sin v\cos 2v + 2e^{2u}\cos v\sin v\, du\,dv$

$= \int_0^{2\pi} (e^2-1)\sin v\cos v\,(2\cos 2v + 1)$

$= \frac{e^2-1}{2}$

(c) $\iint_S \frac{\partial w}{\partial n}\,d\sigma = \iint_S 2xy^2z\,dy\,dz + 2x^2yz\,dz\,dx + x^2y^2\,dx\,dy$

$= \iint_R -2x^2y(1-x^2-y^2)(-2x) - 2x^2y(1-x^2-y^2)(-2y) + x^2y^2$

$= \iint_{x^2+y^2\le 1} 4x^2y^2(1-x^2-y^2) + 4x^2y^2(1-x^2-y^2) + x^2y^2$

$= \iint_{x^2+y^2\le 1} 8x^2y^2(1-x^2-y^2) + x^2y^2$

$= \int_0^{2\pi}\int_0^1 [8r^2\cos^2\theta\, r^2\sin^2\theta(1-r^2) + r^2\cos^2\theta\, r^2\sin^2\theta]\,r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{\sin^2\theta\cos^2\theta}{2}\,d\theta = \frac{\pi}{8}$

(d) $\iint_S \frac{\partial w}{\partial n}\,d\sigma = \iint_S 2x\,dy\,dz - 2y\,dz\,dx + 2z\,dx\,dy$

$= \int_0^{\frac{\pi}{2}}\int_0^1 2e^u\cos v\, e^u\sin v\cos 2v - 2e^u\sin v(-e^u\cos v\cos 2v) + 2\cos v\sin v\cdot e^{2u})\,du\,dv$

$= \int_0^{2\pi}\int_0^1 4e^{2u}\cos v\sin v\cos 2v + 2e^{2u}\cos v\sin v\,du\,dv$

$= \int_0^{2\pi} (e^2-1)\sin v\cos v\,(2\cos 2v+1)$

$= \frac{e^2-1}{2}$

(e) $\operatorname{curl} u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xz \end{vmatrix} = [x,\ y-z,\ -2z]$

$$\iint_S (\operatorname{curl} u \cdot n)\, d\sigma = \iint_S x\,dy\,dz + (y-z)\,dz\,dx - 2z\,dx\,dy$$

$$= \iint_R [-x(2) - (y-2x-3y)(3) - 2(2x+3y)]\,dx\,dy$$

$$= \iint_R (-2x - 3y + 6x + 9y - 4x - 6y)\,dy\,dx$$

$$= 0$$

#1. (a) $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_R 3\,dx\,dy\,dz = 3\,\mathrm{Vol}(R) = 4\pi$

(b) $\iint_S V_n\, d\sigma = \iint_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$

$$= \iiint_R (2x+2y+2z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1 \left(\tfrac{1}{2} + y + z\right) dy\,dz$$

$$= 2\int_0^1 (1+z)\,dz$$

$$= 3$$

(c) $\iint_S e^y \cos z\,dy\,dz + e^x \sin z\,dz\,dx + e^x \cos y\,dx\,dy = \iiint_R 0\,dx\,dy\,dz = 0$

(d) $\nabla F = [2x,\ 2y,\ 2z]$

$$\iint_S (\nabla F \cdot n)\, d\sigma = 2\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

$$= 2\iiint_R 3\,dx\,dy\,dz$$

$$= 6\,\mathrm{vol}(R).$$

(e) $\nabla F = [4x,\ -2y,\ -2z]$

$$\iint_S (\nabla F \cdot n)\, d\sigma = \iint_S 4x\,dy\,dz - 2y\,dz\,dx - 2z\,dx\,dy$$

$$= \iiint_R (4-2-2)\,dx\,dy\,dz$$

$$= 0$$

(f) $\nabla F = \left[-\dfrac{2(x-2)}{2[(x-2)^2+y^2+z^2]^{1/4}},\ \dfrac{-2y}{2[(x-2)^2+y^2+z^2)^{1/4}}\ -\dfrac{2z}{2[(x-2)^2+y^2+z^2]^{1/4}}\right]$

$$\iint_S \nabla F \cdot n\, d\sigma = -\iiint_R \left[\frac{[(x-2)^2+y^2+z^2)^{1/4} - 1/4[(x-2)^2+y^2+z^2]^{-3/4} 2(x-2)(x-2)\,[(x-2)^2+y^2+z^2]^{1/4}}{((x-2)^2+y^2+z^2)^{1/2}} \right.$$

$$+ \iiint_R \frac{z\,\frac{1}{4}[(x-2)^2+y^2+z^2)^{-3/4}(2z)}{((x-2)^2+y^2+z^2)^{1/2}}$$

$$= \iiint_R 0\,dx\,dy\,dz = 0$$

#2. (a) $\iint_S x\,dy\,dz = \iiint_R dx\,dy\,dz = \mathrm{Vol}(R)$, $\iint_S y\,dz\,dx = \mathrm{Vol}(R)$, $\iint z\,dx\,dy = \mathrm{Vol}$

$$\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_R 3\,dx\,dy\,dz = 3\,\mathrm{Vol}(R) = 3V.$$

(b) $\iint_S x^2\,dy\,dz + 2xy\,dz\,dx + 2xz\,dx\,dy = \iiint_R (2x + 2x + 2x)\,dx\,dy\,dz$

$= 6\iiint_R x\,dx\,dy\,dz$

$= 6V\bar{x}$

(c) $\iint_S \text{curl}\,v \cdot n\,d\sigma = \iiint_R \text{div curl}\,v\,dx\,dy\,dz$

$= \iiint_R 0\,dx\,dy\,dz$

$= 0$