5.11 no 4 HW

a) Prove: $\iint_S f \frac{\partial g}{\partial n} d\sigma = \iiint_R f \nabla^2 g + (\nabla f \cdot \nabla g)\, dx\,dy\,dz$

$$\iint_S f \cdot \frac{\partial g}{\partial n} d\sigma = \iint_S (f \nabla g) \cdot \vec{n}\, d\sigma =$$

by def. of directional derivative

by Divergence Theorem

$$= \iiint_R \operatorname{div}(f \nabla g)\, dx\,dy\,dz = \iiint_R \nabla f \cdot \nabla g + f \nabla^2 g\, dx\,dy\,d$$

since $\operatorname{div}(f \cdot \vec{u}) = \nabla f \cdot \vec{u} + f(\nabla \cdot \vec{u})$, $\vec{u} = \nabla g$

b) If g is harmonic, show that $\iint_S \frac{\partial g}{\partial n} d\sigma = 0$

g - harmonic $\Rightarrow \nabla^2 g = 0$

By a) $\iint_S \frac{\partial g}{\partial n} d\sigma \underset{\text{use } f=1}{=} \iiint_R [\underbrace{f \cdot \nabla^2 g}_{0} + \overbrace{\nabla f}^{0} \cdot \nabla g]\, dx\,dy$

$= 0$

c) If f is harmonic $\Rightarrow$

$$\iint_S f \cdot \frac{\partial f}{\partial n} d\sigma = \iiint_R |\nabla f|^2 dx\,dy\,dz$$

f - harmonic $\Rightarrow \nabla^2 f = 0$

By a) with $g = f$ we have

$$\iint_S f \frac{\partial f}{\partial n} d\sigma = \iiint_R [f \underbrace{\nabla^2 f}_{0} + \nabla f \cdot \nabla f] dx dy dz =$$

$$= \iiint_R |\nabla f|^2 dx dy dz$$

d) f - harmonic, $f \equiv 0$ on $S \Rightarrow f \equiv 0$ in R.

by c)

$$0 = \iint_S f \frac{\partial f}{\partial n} d\sigma = \iiint_R |\nabla f|^2 dx dy dz$$

since $f = 0$ on S

$\Rightarrow$ By 4.3 p. 241 since $|\nabla f|^2 \geq 0$ in R, $\iiint_R |\nabla f|^2 dx dy dz$

we have $|\nabla f|^2 \equiv 0$ in R and so

$\nabla f \equiv 0$ in $R \Rightarrow f = \text{const}$ in R

Since $f \equiv 0$ on S, this implies $f \equiv 0$ in R.

f continuous in D

(e) $\int_C U_t \, ds$, $\vec{U} = \text{curl}\,[y^2 i + z^2 j + x^2 k]$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$= [-2z, \ -2x, \ -2y] \qquad \begin{cases} x = 2t+1 \\ y = t^2 \\ z = t^3+1 \end{cases} \quad 0 \le t \le 1$$

$$\int_C U_t \, ds = -2\int_C z\,dx + x\,dy + y\,dz = -2\int_0^1 \{[t^3+1]2 + (2t+1)2t + t^2 \cdot 3t^2\} dt$$

$$= -2\int_0^1 3t^4 + 2t^3 + 4t^2 + 2t + 2 \; dt = -\frac{163}{15}$$

2 (a) $\vec{U} = \text{grad}\, F$

Show $\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} U_t \, ds = F(x_2,y_2,z_2) - F(x_1,y_1,z_1)$

$$\vec{U} = Pi + Qj + Rk, \qquad \nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$

If $\vec{U} = \nabla F$ then $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$, $R = \frac{\partial F}{\partial z}$

So $\int_{A=(x_1,y_1,z_1)}^{B=(x_2,y_2,z_2)} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = \int dF = F(x_2, y_2, z_2) - F(x_1, y_1, z_1)$

(b) $\int_C U_t \, ds = \int_C \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = \int dF = F(x,y,z) - F(x,y,z)$

$= 0$

(because the path is closed)

by f) $(f-g) \equiv \text{const}$ in R

$f \equiv g + \text{const}$ in R

h) If f, g are harmonic in R,

$\frac{\partial f}{\partial n} = -f + h$, $\frac{\partial g}{\partial n} = -g + h$ on S then $\underset{\text{in } R}{f \equiv g}$

f, g - harmonic $\Rightarrow$ $(f-g)$ - harmonic and

$$\frac{\partial (f-g)}{\partial n} = -f + g = -(f-g)$$

By part c) applied to $(f-g)$

$$\iint_S (f-g) \frac{\partial (f-g)}{\partial n} d\sigma = \iiint_R |\nabla (f-g)|^2 dx\,dy\,dz \geq 0$$

$$\parallel$$

$$-\iint_S (f-g)^2 d\sigma \leq 0$$

$\Rightarrow$ all three numbers in the above formula are $= 0$

$$\Rightarrow \iiint_R |\nabla (f-g)|^2 dx\,dy\,dz = 0 \Rightarrow$$

$$|\nabla (f-g)|^2 \equiv 0 \text{ in } R, \quad \nabla (f-g) \underset{\text{in } R}{\equiv} 0 \Rightarrow$$

$\Rightarrow f - g \equiv \text{const}$ in R, $f \equiv g + c$ in R, c - const

$\Rightarrow f - g \equiv c$ in R

$$\Rightarrow 0 = \iint_S (f-g)^2 d\sigma = \iint_S c^2 d\sigma = c^2 \cdot \text{Area}(S)$$

$$\Rightarrow c^2 = 0, \quad c = 0 \Rightarrow f - g \equiv 0 \text{ in } R$$

$$f \equiv g \text{ in } R.$$

i) Suppose f, g satisfy $\nabla^2 f = -4\pi h$

$\nabla^2 g = -4\pi h$

and $f \equiv g$ on $S \Rightarrow f \equiv g$ in R

$\nabla^2 f = -4\pi h$, $\nabla^2 g = -4\pi h \Rightarrow \nabla^2 (f-g) = 0$

that is $(f-g)$ – harmonic

$f \equiv g$ on $S \Rightarrow (f-g) \equiv 0$ on $S \Rightarrow$ by d)

$(f-g) \equiv 0$ in R, $f \equiv g$ in R

j) Prove $\displaystyle\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d\sigma = \iiint_R (f\nabla^2 g - g\nabla^2 f)\, dx$

By a) $\displaystyle\iint_S f\frac{\partial g}{\partial n}\, d\sigma = \iiint_R (f\nabla^2 g - \nabla f \cdot \nabla g)\, dx\,dy\,dz$

$\displaystyle\iint_S g\frac{\partial f}{\partial n}\, d\sigma = \iiint_R (g\nabla^2 f - \nabla f \cdot \nabla g)\, dx\,dy\,dz \underset{\text{substra}}{\Rightarrow}$

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d\sigma = \iiint_R (f\nabla^2 g - g\nabla^2 f)\, dx\,dy\,dz$$

k) Suppose f, g - harmonic $\Rightarrow$ By j)

$$\iint\limits_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\right) d\sigma = \iiint\limits_R \left(f \cdot \underbrace{\nabla^2 g}_{0} - g \underbrace{\nabla^2 f}_{0}\right) dx\,dy\,dz =$$

l) Suppose $\nabla^2 f = hf$, $\nabla^2 g = hg$ $\Rightarrow$ By j)

$$\iint\limits_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\right) d\sigma = \iiint\limits_R \left(f \nabla^2 g - g \nabla^2 f\right) dx\,dy\,dz =$$

$$= \iiint\limits_R \underbrace{(hfg - ghf)}_{0} dx\,dy\,dz = 0$$

<u>no 5 ch 5.12</u>

a) $\iint\limits_S f(\vec{n} \cdot \vec{i}) d\sigma = \iiint\limits_R \frac{\partial f}{\partial x} dV$

<u>We have:</u>

$$\iint\limits_S f \cdot (\vec{n} \cdot \vec{i}) d\sigma =$$

$$= \iint\limits_S (f \vec{i}) \vec{n} d\sigma \overset{\text{Divergence theorem}}{=} \iiint\limits_R \operatorname{div}(f \vec{i}) dV =$$

$$= \iiint\limits_R \frac{\partial f}{\partial x} dV$$

b) $$\iint_S f\vec{n}\,d\sigma = \iiint_R \vec{\nabla f}\,dV$$

By definition:

$$\iiint_S f\vec{n}\,d\sigma = \Big[\iint_S f(\vec{n}\cdot\vec{i}\,)\,d\sigma,\ \iint_S f(\vec{n}\cdot\vec{j}\,)\,d\sigma,\ \iint_S f(\vec{n}\cdot\vec{k}\,)$$

$\qquad\qquad\qquad\qquad \parallel \to$ by a) $\qquad\qquad \parallel \to$ by a) $\qquad\qquad \parallel \to$ by

$$\iiint_R \vec{\nabla f}\,dV = \Big[\iiint_R \frac{\partial f}{\partial x}\,dV,\ \iiint_R \frac{\partial f}{\partial y}\,dV,\ \iiint_R \frac{\partial f}{\partial z}\,dV\Big]$$

c) $$\iint_S (\vec{V}\times\vec{i}\,)\cdot\vec{n}\,d\sigma = \iiint_R (\operatorname{curl}\vec{V})\cdot\vec{i}\,dV$$

$$\iint_S (\vec{V}\times\vec{i}\,)\cdot\vec{n}\,d\sigma \underset{\text{Divergence thm}}{=} \iiint_R \operatorname{div}(\vec{V}\times\vec{i}\,)\,dV$$

$$\operatorname{div}(\vec{V}\times\vec{i}\,) = \cancel{} = \text{by } (3.35)$$

$$= \vec{i}\cdot\operatorname{curl}\vec{V} - \vec{V}\cdot\underbrace{\operatorname{curl}\vec{i}}_{0\ \to\ \text{since } \vec{i} \text{ is constant}} = \vec{i}\cdot\operatorname{curl}\vec{V} \quad \Rightarrow$$

$$\iiint_R \operatorname{div}(\vec{V}\times\vec{i}\,)\,dV = \iiint_R (\operatorname{curl}\vec{V})\cdot\vec{i}\,dV$$

as required

d) $\iint_S (\vec{n} \times \vec{v})\, d\sigma = \iiint_R \operatorname{curl} \vec{v}\, dV$

$\vec{n} \times \vec{v} = [\vec{n} \cdot (\vec{v} \times \vec{i}), \vec{n} \bullet (\vec{v} \times \vec{j}), \vec{n} \bullet (\vec{v} \times \vec{k})]$

see (1.34)

$\operatorname{curl} \vec{v} = [(\operatorname{curl} \vec{v}) \cdot \vec{i}, (\operatorname{curl} \vec{v}) \cdot \vec{j}, (\operatorname{curl} \vec{v}) \cdot \vec{k}]$

$\Rightarrow$ by c)

$[\iint_S \vec{n} \cdot (\vec{v} \times \vec{i})\, d\sigma, \iint_S \vec{n} \cdot (\vec{v} \times \vec{j})\, d\sigma, \iint_S \vec{n} \cdot (\vec{v} \times \vec{k})\, d\sigma$

$\parallel \qquad \parallel \qquad \parallel$

$[\iiint_R (\operatorname{curl} \vec{v}) \cdot \vec{i}\, dV, \iiint_R (\operatorname{curl} \vec{v}) \cdot \vec{j}\, dV, \iiint_R (\operatorname{curl} \vec{v}) \cdot \vec{k}\, d$

5.13 no 1

a) $\oint_C u_T\, ds \qquad C: x^2 + y^2 = 1, \; z = 2$

$\vec{u} = -3y\vec{i} + 3x\vec{j} + \vec{k}$

By Stoke's theorem

$\oint_C u_T\, ds = \iint_S \operatorname{div} \vec{u}\, d\sigma = \iint_S (0 + 0 + 0)\, d\sigma = 0$

b) $\oint_C \underbrace{2xy^2z}_{L}\,dx + \underbrace{2x^2yz}_{M}\,dy + \underbrace{(x^2y^2 - 2z)}_{N}\,dz =$

$= \iint_S \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\right) dy\,dz + \left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}\right) dz\,dx + \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx$

$= \iint_S \underbrace{(2yx^2 - 2x^2y)}_{0}\,dy\,dz + \underbrace{(2xy^2 - 2xy^2)}_{0}\,dz\,dx + \underbrace{(4xyz - 4xyz)}_{0}\,dx\,dy$

$= 0$