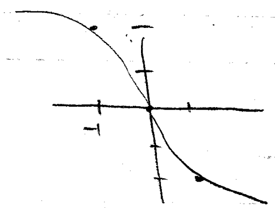
H/wk 5

1. (a) $y = x^3 - 3x$



$y' = 3(x^2 - 1)$

$\therefore x = \pm 1$

$\therefore$ maximum at -1, minimum at 1

(b) $y = 2\sin x + \sin 2x$

$y' = 2\cos x + 2\cos 2x$

$2(\cos x + \cos 2x) = 0$

$\therefore$ maximum at $\pi/3 + 2n\pi$ minimum at $-\pi/3 + 2n\pi$ horizontal infl. $\pi + 2n\pi$

(c) $y = e^{-x} - e^{-2x}$

$y' = -e^{-x} + 2e^{-2x}$

let $e^{-x} = A$, $2A^2 - A = 0$ $A(2A-1) = 0$ $A = 0 \cdot A = \frac{1}{2}$

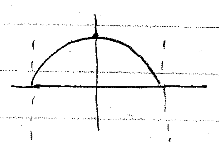
$e^{-x} = 0$ (X) or $e^{-x} = \frac{1}{2}$ $\therefore x = \ln 2$; maximum.

2. $y = x^n$ at $x = 0$

$y' = nx^{n-1}$

$\therefore$ minimum $n = 2, 4, 6 \ldots$ horizontal for $n = 3, 5, n \ldots$

3. (a) $y = \cos x$ $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

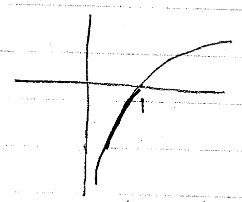


$y' = -\sin x$

$\therefore x = 0$ max, So $y = \cos 0 = 1$ maximum

$y = \cos \frac{\pi}{2} = 0$ minimum.

(b) $y = \log x \qquad 0 < x \leq 1$



$y' = \dfrac{1}{x}$

$\therefore\ y = 0$; maximum   not exist minimum

(c) $y = \tanh x \qquad$ all x

$y' = \operatorname{sech}^2 > 0$ there are no max. and min.

(d) $y = \dfrac{x}{1+x^2} \qquad$ all x

$$y' = \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

so $1 - x^2 = 0 \qquad x = \pm 1.$ $\therefore\ y = \frac{1}{2}$; maximum $y = -\frac{1}{2}$; minimum

By P156.

4. (a) $z = \sqrt{1-x^2-y^2} \qquad \dfrac{\partial z}{\partial x} = -x \Rightarrow x = 0 \qquad \dfrac{\partial z}{\partial y} = -y \Rightarrow y = 0$

$A = -1 \qquad B = 0 \qquad C = -1$

$\therefore\ B^2 - AC < 0 \qquad A + C < 0$ $\therefore$ maximum at $(0,0)$

(b) $z = 1 + x^2 + y^2 \qquad \dfrac{\partial z}{\partial x} = 2x = 0 \Rightarrow x = 0 \qquad \dfrac{\partial z}{\partial y} = 2y = 0 \Rightarrow y = 0$

$A = \dfrac{\partial^2 z}{\partial x^2} = 2 \qquad B = 0 \qquad C = 2$

$\therefore\ B^2 - AC < 0 \qquad A + C > 0$ $\therefore$ minimum at $(0,0)$

(c) $z = 2x^2 - xy - 3y^2 - 3x + 7y$

$\frac{\partial z}{\partial x} = 4x - y - 3 \Rightarrow 4x - y - 3 = 0$

$\frac{\partial z}{\partial y} = -x - 6y + 7 \Rightarrow -x - 6y + 7 = 0$

$\Bigg\} \Rightarrow x = 1 \quad y = 1$

$\frac{\partial^2 z}{\partial x^2} = 4 \qquad \frac{\partial^2 z}{\partial y^2} = -6 \qquad \frac{\partial^2 z}{\partial x \partial y} = -1$

$\therefore B^2 - AC = (-1)^2 - (4)(-6) = 25 > 0 \qquad A + C = -2 < 0 \qquad \therefore$ Saddle at $(1,1)$

(d) $z = x^2 - 5xy - y^2$

$\frac{\partial z}{\partial x} = 2x - 5y \Rightarrow 2x - 5y = 0$

$\frac{\partial z}{\partial y} = -5x - 2y \Rightarrow -5x - 2y = 0$

$\Bigg\} \Rightarrow x = 0 \quad y = 0$

$A = 2 \qquad B = -5 \qquad C = -2$

$\therefore B^2 - AC > 0 \qquad \therefore$ Saddle at $(0,0)$

(e) $z = x^2 - 2xy + y^2 = (x-y)^2 \geq 0$

$\frac{\partial z}{\partial x} = 2x - 2y \Rightarrow 2x - 2y = 0$

$\frac{\partial z}{\partial y} = -2x + 2y \Rightarrow -2x + 2y = 0$

$\Bigg\} \Rightarrow x = y$

$A = 2 \qquad B = -2 \qquad C = 2$

$x = y \Rightarrow z(x,y) = 0$

$z \geq 0$ always $\Rightarrow$

$\therefore B^2 - AC = 0 \qquad \therefore$ critical point at every point of $y = x$ and a relative minimum

(f) $z = x^3 - 3xy^2 + y^3$

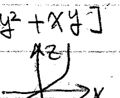
$\frac{\partial z}{\partial x} = 3x^2 - 3y^2 \Rightarrow x^2 - y^2 = 0$

$\frac{\partial z}{\partial y} = -6xy + 3y^2 \Rightarrow xy - y^2 = 0$

$\Bigg\} \quad x = 0 \quad y = 0$

$A = 6x \qquad B = -6y \qquad C = -6x + 6y \quad | \quad x = y = 0 \Rightarrow A = B = C = 0$

$B^2 - AC = 36[x^2 + y^2 + xy] = 36\left[(x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2\right] = 0$

$y = 0 \Rightarrow z = x^3$  $\Rightarrow (0,0)$ - not a relative extremum

(g) $z = x^2 - 2x(\sin y + \cos y) + 1$

$\frac{\partial z}{\partial x} = 2x - 2(\sin y + \cos y)$

$\frac{\partial z}{\partial y} = -2x[\cos y - \sin y]$

$\Rightarrow$ ① $x=0$, $\sin y + \cos y = 0$, i.e. $(0, n\pi - \frac{\pi}{4})$

$A = 2 \quad B = -2(\cos y - \sin y) \quad C = +2x[\sin y + \cos y]$

$B^2 - AC = 4(\cos y - \sin y)^2 - 4x(\sin y + \cos y)$

② $\cos y = \sin y$, $x = 2\sin y$

$(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ $(-\sqrt{2}, \frac{5}{4}\pi + 2n\pi)$

$\therefore$ minimum at $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ $(-\sqrt{2}, \frac{5}{4}\pi + 2n\pi)$ but nothing at $(0, n\pi - \frac{\pi}{4})$

(h) $z = xy^2 + x^2y - xy$

$\frac{\partial z}{\partial x} = y^2 + 2xy - y$

$\frac{\partial z}{\partial y} = 2xy + x^2 - x$

$\Rightarrow$ $y(y + 2x - 1) = 0$, $x(2y + x - 1) = 0$

$A = y \quad B = 2y + 2x - 1 \quad C = 2x$

$B^2 - AC = (2y + 2x - 1)^2 - 2xy$

$\Downarrow$ $y = 0$: $x = 0$ $(0,0)$; $x = 1$ $(1,0)$

$y = 1 - 2x \Rightarrow$ $x = 0$ $(0,1)$; $x = \frac{1}{3}$ $(\frac{1}{3}, \frac{1}{3})$

$\therefore$ minimum at $(\frac{1}{3}, \frac{1}{3})$

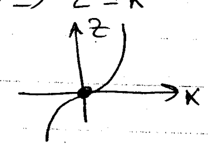
saddle at $(0,0)$ $(1,0)$ $(0,1)$

(i) $z = x^3 + y^3$

$\frac{\partial z}{\partial x} = 3x^2 \Rightarrow x = 0$

$\frac{\partial z}{\partial y} = 3y^2 \Rightarrow y = 0$

$y = 0 \Rightarrow z = K^3$



$A = 6x \quad B = 0 \quad C = 6y = 0$

$B^2 - AC = -36xy = 0$ $\therefore$ neither maximum nor minimum.

(j) $z = x^4 + 3x^2y^2 + y^4$

$Z \geq 0$ for any x, y

$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2 = 0 \Rightarrow$

$\frac{\partial z}{\partial y} = 6x^2y + 4y^3 = 0 \Rightarrow$

$x = 0, y = 0$ – the only critical point

$Z(0,0) = 0$, $Z \geq 0$ always $\Rightarrow$ $(0,0)$ – relative minimum

(k) $z = [x^2+(y+1)^2][x^2+(y-1)^2] = [x^2+2y+y^2+1][x^2-2y+y^2+1]$

$$\frac{\partial z}{\partial x} = 2x\cdot[x^2+y^2-2y+1]+[x^2+2y+y^2+1]\cdot 2x =$$

$$= 2x[2x^2+2y^2+2] = 4x[x^2+y^2+1]$$

$$\frac{\partial z}{\partial y} = (2y+2)[x^2-2y+y^2+1]+[x^2+2y+y^2+1][2y-2] =$$

$$= 2yx^2 \cancel{+2x^2} \cancel{-4y^2} \cancel{-4y} + 2y^3 \cancel{+2y^2} \cancel{+2y} \cancel{+2} +$$

$$+ 2x^2y \cancel{-2x^2} \cancel{+4y^2} - 4y + 2y^3 \cancel{-2y^2} \cancel{+2y} \cancel{-2}$$

$$= 2yx^2+2y^3+2x^2y-4y+2y^3 =$$

$$= 4y^3+4yx^2-4y = 4y(y^2-x^2-1)$$

$$\begin{cases} 4x[x^2+y^2+1]=0 \\ 4y(y^2-x^2-1)=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ 4y(y^2-1)=0 \end{cases} \quad \begin{cases} x=0 \\ y=0,\pm 1 \end{cases}$$

$(0,0)$, $(0,1)$, $(0,-1)$ – critical points

$$\frac{\partial^2 z}{\partial x^2} = 12x^2+4y^2+4 \qquad \frac{\partial^2 z}{\partial x \partial y} = 8xy$$

$$\frac{\partial^2 z}{\partial y^2} = 12y^2-4x^2-4$$

$(0,0) \Rightarrow A=4,\ C=-4,\ B=0 \qquad B^2-AC=16>0$ – saddle point

$(0,1) \Rightarrow A=8 \quad C=8 \quad B=0 \qquad B^2-AC=-64<0$ – rel. min

$(0,-1) \Rightarrow A=8 \quad C=8 \quad B=0 \qquad B^2-AC=-64<0$ – rel. min.