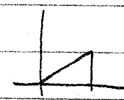
1 (a) $\iint_R (x^2+y^2)\,dx\,dy$



$0 \le x \le 1$

$0 \le y \le x$

$= \int_0^1 \int_0^x (x^2+y^2)\,dy\,dx = \int_0^1 x^2y + \frac{y^3}{3}\Big|_0^x = \int_0^1 \frac{x^3}{3} + x^3\,dy = \frac{1}{3}$

(b) $\iiint_R u^2v^2w\,du\,dv\,dw$ $\quad u^2+v^2 \le 1 \quad 0 \le w \le 1$

let $u^2+v^2=r^2 \quad r\sin\theta = v \quad r\cos\theta = u$

Then $0 \le r \le 1 \quad 0 \le \theta \le 2\pi \quad 0 \le w \le 1$

$\int_0^1\int_0^{2\pi}\int_0^1 (r^2\sin^2\theta)(r^2\cos^2\theta)\,w\,r\,dr\,d\theta\,dw$

$= 4\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \left(r^2\tfrac{1}{2}(1-\cos 2\theta)\right)\left(r^2\tfrac{1}{2}(1+\cos 2\theta)\right) rw\,dr\,d\theta\,dw$

$= 4\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \frac{r^4}{4}(1-\cos^2 2\theta)\,rw\,dr\,d\theta\,dw$

$= 4\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \frac{r^5}{4}\left(1-\tfrac{1}{2}(1+\cos 4\theta)\right) w\,dr\,d\theta\,dw$

$= 4\int_0^1 \int_0^{\frac{\pi}{2}} w\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\right)\left(\tfrac{1}{24}\right) d\theta\,dw$

$= \frac{4}{48}\int_0^1 \int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,w\,d\theta\,dw$

$= \frac{1}{12}\int_0^1 w\left[\theta - \frac{\sin 4\theta}{4}\right]\Big|_0^{\frac{\pi}{2}} dw$

$= \frac{1}{12}\int_0^1 w\left[\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right] dw$

$= \frac{1}{12}\int_0^1 \frac{\pi}{2} w\,dw = \frac{\pi}{48}$

(c) $\iint_R r^3 \cos\theta \, dr\, d\theta \qquad 1 \le r \le 2 \quad \frac{\pi}{4} \le \theta \le \pi$

$\shortparallel$

$\int_1^2 \int_{\frac{\pi}{4}}^{\pi} r^3 \cos\theta \, d\theta\, dr = \int_{\frac{\pi}{4}}^{\pi} \frac{r^4}{4} \cos\theta \Big|_1^2 d\theta = \int_{\frac{\pi}{4}}^{\pi} 4\cos\theta - \frac{1}{4}\cos\theta \, d\theta$

$= \frac{-15\sqrt{2}}{8}$

(d) $\iiint_R x+z \, dx\, dy\, dz$

$\int_0^3 \int_0^{1-\frac{z}{3}} \int_0^{2-\frac{2}{3}z-2x} (x+z) \, dy\, dx\, dz$

$= \int_0^3 \int_0^{1-\frac{z}{3}} y(x+z) \Big|_0^{2-\frac{2}{3}z-2x} = \int_0^3 \int_0^{1-\frac{z}{3}} \frac{-2(x+z)(3x+z-3)}{3}$

$= \int_0^3 \frac{-x(2x^2+4xz-3x+2z^2-6z)}{3} \Big|_0^{1-\frac{z}{3}}$

$= \int_0^3 \frac{(z-3)^2(8z+8)}{81}$

$= \left[\frac{z(2z^3-15z^2+27z+27)}{8} \right] \Big|_0^3 = 1$

2. (a) $z = e^x \cos y, \quad 0 \le x \le 1 \quad 0 \le y \le \frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \int_0^1 e^x \cos y \, dx\, dy = \int_0^{\frac{\pi}{2}} e^x \cos y \Big|_0^1 dy$

$= \int_0^{\frac{\pi}{2}} (e\cos y - \cos y) \, dy = e\sin y - \sin y \Big|_0^{\frac{\pi}{2}} = e - 1$

(b) $z = x^2 e^{-x-y} \quad 0 \le x \le 1 \quad 0 \le y \le 2$

$$\int_0^2 \int_0^1 x^2 e^{-x-y}\, dx\, dy = \int_0^2 -(x^2+2x+2)e^{-x-y} \Big|_0^1 dy$$

$$= \int_0^2 -5e^{-1-y} + 2e^{-y}\, dy$$

$$= \left(\frac{-5e^{-1-y}}{-1} + \frac{2e^{-y}}{-1} \right) \Big|_0^2$$

$$= 5e^{-3} - 2e^{-2} - 5e^{-1} + 2.$$

(c) $z = x^2 y \quad 0 \le x \le 1 \quad x+1 \le y \le x+2$

$$\int_0^1 \int_{x+1}^{x+2} x^2 y\, dy\, dx = \int_0^1 \frac{x^2}{2}(x^2+4x+4) - \frac{x^2}{2}(x^2+2x+1)\, dx$$

$$= \int_0^1 x^3 + \frac{3}{2}x^2\, dx$$

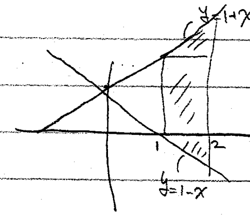
$$= \left(\frac{x^2}{4} + \frac{x^3}{2} \right) \Big|_0^1 = \frac{3}{4}$$

(d) $z = \sqrt{x^2 - y^2} \quad x^2 - y^2 \ge 0 \quad 0 \le x \le 1$

$$\int_0^1 \int_{-x}^{x} \sqrt{x^2-y^2} = \int_0^1 \sin^{-1}\left(y \cdot \frac{1}{|x|}\right) \cdot x^2/2 + y\sqrt{x^2-y^2}/2 \Big|_{-x}^{x}$$

$$= \int_0^1 \frac{\pi}{4}x^2 + \frac{\pi}{4}x^2 = \int_0^1 \frac{\pi x^2}{2} = \frac{\pi}{6}$$

3 (a) $1 \le x \le 2 \qquad 1-x \le y \le 1+x$



$y-1 \le x \le 2,\ 2 \le y \le 3$

$1 \le x \le 2,\ 0 \le y \le$

$1-y \le x \le 2,\ -1 \le y$

$$\int_1^2 \int_{1-x}^{1+x} f(x,y)\,dy\,dx$$

$$\int_2^3 \int_{y-1}^2 f(x,y)\,dx\,dy + \int_0^2 \int_1^2 f(x,y)\,dx\,dy + \int_{-1}^0 \int_{1-y}^2 f(x,y)\,dx\,dy.$$

(b) $\cancel{y^2 + x(x-1) \le 0}$

$y^2 + x^2 - x \le 0$

$\Downarrow$

$y^2 + (x-\frac{1}{2})^2 \le \frac{1}{4}$

$y^2 \le x - x^2 \quad \rightarrow \quad -\sqrt{x-x^2} \le y \le \sqrt{x-x^2}$

$0 \le x \le 1$

$(x-\frac{1}{2})^2 \le \frac{1}{4} - y^2 \rightarrow \quad -\sqrt{\frac{1}{4}-y^2} \le x - \frac{1}{2} \le \sqrt{\frac{1}{4}-y^2}$

$\frac{1}{2} - \sqrt{\frac{1}{4}-y^2} \le x \le \frac{1}{2} + \sqrt{\frac{1}{4}-y^2}$

$-\frac{1}{2} \le y \le \frac{1}{2}$

$$\therefore \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} f(x,y)\,dy\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-y^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}+y^2}} f(x,y)\,dx\,dy$$

4 (a) $\int_0^1 \int_0^1 \int_0^1 \sqrt{x+y+z}\, dx\, dy\, dz = \int_0^1 \int_0^1 \left[\frac{2}{3}(x+y+z)^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}\int_0^1 \int_0^1 \left[(1+y+z)^{3/2} - (y+z)^{3/2}\right] dy\, dz$

$= \frac{4}{15}\int_0^1 \left[(1+y+z)^{5/2} - (y+z)^{5/2}\right]_0^1 dz$

$= \frac{4}{15}\int_0^1 \left[(z+z)^{5/2} - (1+z)^{5/2} - (1+z)^{5/2} + z^{5/2}\right] dz$

$= \frac{4}{15}\int_0^1 \left[(z+z)^{5/2} - 2(1+z)^{5/2} + z^{5/2}\right] dz$

$= \frac{8}{105}\left[(z+z)^{7/2} - 2(1+z)^{7/2} + z^{7/2}\right]_0^1$

$= \frac{8}{35}(9\sqrt{3} - 8\sqrt{2} + 1)$

(b) $\iiint_R (x^2+z^2)\, dx\, dy\, dz$

$\begin{cases} 0 \le z \le 1 \\ z-1 \le x \le 1-z \\ z-1 \le y \le 1-z \end{cases}$

$\int_0^1 \int_{z-1}^{1-z} \int_{z-1}^{1-z} (x^2+z^2)\, dx\, dy\, dz = \int_0^1 \int_{z-1}^{1-z} \left[\frac{x^3}{3} + z^2 x\right]_{z-1}^{1-z} dy\, dz$

$= \int_0^1 \int_{z-1}^{1-z} \left[\frac{(1-z)^3}{3} + z^2(1-z) - \frac{(z-1)^3}{3} - z^2(z-1)\right]$

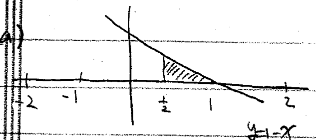
$= \int_0^1 \int_{z-1}^{1-z} \left(\frac{2}{3}(1-z)^3 + 2z^2(1-z)\right) dy\, dz$

$= \int_0^1 \frac{4}{3}(1-z)^4 + 4z^2(1-z^2)\, dz$

$= \left[-\frac{4}{15}(1-z)^5 + \frac{4}{3}z^3 - 2z^4 + \frac{4}{5}z^5\right]_0^1$

$= \frac{2}{?}$

5(a)

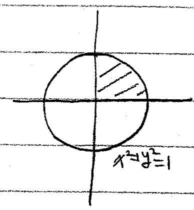


$0 \le y \le 1-x$

$\frac{1}{2} \le x \le 1$

$$\int_{\frac{1}{2}}^{1}\int_{0}^{1-x} f(x,y)\,dy\,dx = \int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1-y} f(x,y)\,dx\,dy$$

(b)

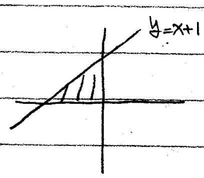


$0 \le y \le \sqrt{1-x^2}$

$0 \le x \le 1$

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

(c)

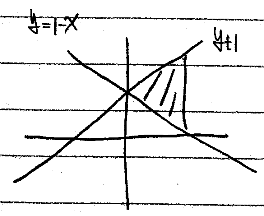


$y-1 \le x \le 0$

$0 \le y \le 1$

$$\int_{0}^{1}\int_{y-1}^{0} f(x,y)\,dx\,dy = \int_{-1}^{0}\int_{0}^{x+1} f(x,y)\,dy\,dx$$

(d)



$1-x \le y \le 1+x$

$0 \le x \le 1$

$$\int_{0}^{1}\int_{1-x}^{1+x} f(x,y)\,dy\,dx = \int_{0}^{2}\int_{|y-1|}^{1} f(x,y)\,dx\,dy.$$