Math 285 Section F1 Exam 1 (WITH SOLUTIONS)
Prof. I.Kapovich September 15, 2000

You may use a calculator and one 8"x11" two-sided sheet of formulas. No textbooks or lecture notes are allowed during the exam. PLEASE PRINT YOUR NAME AND YOUR NETID ON YOUR EXAM.

Theory

Problem 1.[15 points] Select the correct answer for each of the following questions. (Each question has exactly one correct answer)

(1) For the initial value problem $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(1) = 4$ on the interval $x > 0$

(a) A solution exists but is not guaranteed to be unique.
(b) A solution is guaranteed to exist on a smaller subinterval of the interval $x > 0$ but not necessarily on the interval $x > 0$.
(c) A unique solution is guaranteed to exist on the interval $x > 0$.
(d) A solution is not guaranteed to exist on the interval $x > 0$.

Answer: (c)

(2) Any solution $y = y(x)$ of the differential equation $y'' = x^2 + y^2 + 1$ is

(a) Concave down on any open subinterval of the domain of $y(x)$.
(b) Concave up on any open subinterval of the domain of $y(x)$.
(c) May be either concave up or concave down on an open subinterval of the domain of $y(x)$.

Answer: (b)

(3) Making the substitution $v = ax + by + c$ in an equation $y' = F(ax + by + c)$ transforms this differential equation into

(a) a linear first order differential equation
(b) a separable differential equation
(c) an exact equation
(d) a homogeneous equation

Answer: (b)

(4) The isoclines of the differential equation $y' = x^2 + y^2$

(a) are straight lines
(b) are hyperbolas
(c) are circles centered at the origin
(d) do not exist

Answer: (c)

Techniques and computation

Problem 2.[15 points] Select the correct answer for each of the following questions. (Each question has exactly one correct answer).

(1) The differential equation $y^4x + 2y'e^x = 0$ is

(a) linear
(b) separable
(c) homogeneous
(d) exact

Answer: (b)

(2) The differential equation $x^2y + y^3 = 2xy^2y'$ is

(a) linear

(b) separable
(c) homogeneous
(d) exact

Answer: (c)

(3) The differential equation $xy' - y\sin x + y^5 = 0$ is

(a) linear
(b) Bernoulli
(c) homogeneous
(d) exact

Answer: (b)

Problem 3.[20 points] Solve the initial value problem $y' - y = e^{3x}$, $y(0) = 1$. Give all the details of your work.

Solution. This is a 1-st order linear equation. First we compute the integrating factor:

$$\rho(x) = e^{\int -dx} = e^{-x}.$$

Multiplying the main equation by the integrating factor we get:

$$y'e^{-x} - ye^{-x} = \frac{d}{dx}(ye^{-x}) = e^{3x}e^{-x} = e^{2x}.$$

Hence

$$ye^{-x} = \int e^{2x}\,dx = (1/2)e^{2x} + C \quad \text{ and so } y = (1/2)e^{3x} + Ce^{x}.$$

Since $y(0) = 1$ we have:

$$(1/2)e^0 + Ce^0 = 1 \text{ and therefore } C = 1/2$$

Thus $y = (1/2)e^3x + (1/2)e^x$.

Problem 4.[20 points] Find the general solution of the differential equation $y' = (x + y - 7)^2$. Give all the details of your work.

Solution. Since the equation is of the form $y' = F(ax + by + c)$, we will make the substitution $v = x + y - 7$ to get a separable equation.

For $v = x + y - 7$ we have $\frac{dv}{dx} = 1 + \frac{dy}{dx}$, and so $\frac{dy}{dx} = \frac{dv}{dx} - 1$. Therefore our main equation $y' = (x + y - 7)^2$ can be re-written as:

$$\begin{aligned}
\frac{dv}{dx} - 1 &= v^2 \\
\frac{dv}{dx} &= v^2 + 1 \\
\frac{dv}{v^2 + 1} &= dx \\
\int \frac{dv}{v^2 + 1} &= \int dx \\
\arctan v &= x + C \\
v &= \tan(x + C) \\
x + y - 7 &= \tan(x + C)
\end{aligned}$$

$$y = \tan(x + C) - x + 7.$$

Thus $y = \tan(x + C) - x + 7$ is the general solution of the equation $y' = (x + y - 7)^2$.

Thinking

Problem 5.[30 points] A tank in the form of a cube with side-length 9 feet is completely filled with gasoline. Then a small hole is opened at the bottom. After 20 minutes the level of gasoline in the tank drops to 4 feet.

(a) Find the function $y(t)$, where $y(t)$ is the level of gasoline in the tank (in feet) after t minutes. Give all the details of your work.

(b) Determine when the tank becomes empty. Give all the details of your work.

Hint: You must use Toricelli's law here. Even if it may seem so from the first glance, it is NOT TRUE that the level of gasoline in the tank is decreasing at a constant rate.

Solution.

(a) Recall that $y(t)$ is the level of water in the tank in (in feet) after t minutes. Since the tank is a cube with side-length 9 feet, any cross-section of the tank is a square with side-length 9 feet. Therefore the area of the cross-section of the tank at level y feet is $A(y) = 9 \cdot 9 = 81 ft^2$. Thus the Torricelli law gives us the following separable differential equation:

$$\begin{aligned}
81\frac{dy}{dt} &= -k\sqrt{y} \\
y^{-1/2}dy &= -k_1\, dt \text{ where } k_1 = k/81 \\
\int y^{-1/2}dy &= \int -k_1\, dt \\
2y^{1/2} &= -k_1 t + C \\
y^{1/2} &= \frac{C - k_1 t}{2} \\
y &= \frac{(C - k_1 t)^2}{4}.
\end{aligned}$$

Since the tank is full at $t = 0$, we have $y(0) = 9$. Substituting $t = 0, y = 9$ in $2y^{1/2} = -k_1 t + C$ we get $C = 2 \cdot 9^{1/2} = 6$. Since $y(20) = 4$, substituting $t = 20$, $y = 4$ in $2y^{1/2} = -k_1 t + 6$ we get $2 \cdot 4^{1/2} = -k_1 \cdot 20 + 6$. So $20k_1 = 6 - 4 = 2$ and $k_1 = 2/20 = 1/10$. Thus

$$y = \frac{(C - k_1 t)^2}{4} = \frac{(6 - \frac{1}{10}t)^2}{4} = \frac{(60 - t)^2}{400}.$$

(b) The tank is empty when $y(t) = 0$. By solving

$$\frac{(60 - t)^2}{400} = 0$$

we get $60 - t = 0$ and $t = 60$ minutes.