Math 285 Section F1 Exam 2 (WITH SOLUTIONS)
Prof. I.Kapovich October 20, 2000

Theory

Problem 1.[15 points] Select the correct answer for each of the following questions. (Each question has exactly one correct answer)

(1) Suppose $p(x)$ and $q(x)$ are continuous on the entire real line. Consider the problem

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(0) = a, y'(0) = b \end{cases}$$

(a) For any numbers a, b a solution exists but is not guaranteed to be unique.
(b) There are some a, b such that a solution does not exists.
(c) For $a = 1$, $b = 1$ there exist two different solutions.
(d) For $a = 0$, $b = 0$ the only solution is $y = 0$.

Answer: D

(2) Let $f(x)$, $g(x)$ be two functions which are continuous and differentiable on the entire real line. Let $W = W(f, g)$ be their Wronskian.

(a) If $W(0) = 0$ then the functions $f(x), g(x)$ are linearly dependent on $(-\infty, \infty)$.
(b) It is possible that $W(0) \neq 0$ and the functions $f(x), g(x)$ are linearly dependent on $(-\infty, \infty)$.
(c) If $f(x), g(x)$ are linearly dependent on $(-\infty, \infty)$ then $W(1) + W(2) = 0$.

Answer: C

(3) Suppose the characteristic equation of a linear homogeneous differential equation of degree $2n$ with constant coefficients is $(r^2 + \alpha^2)^n = 0$ where $\alpha > 0$ is some real number.

(a) The functions $e^{\alpha x}, xe^{\alpha x}, \ldots, x^{n-1}e^{\alpha x}$ are solutions of our differential equation.
(b) This differential equation cannot have more than n linearly independent solutions.
(c) The function $y = 2\cos(\alpha x) + 7x^{n-1}\sin(\alpha x)$ is a solution of our differential equation.
(d) The only solution of this differential equation is $y = 0$.

Answer: C

Techniques and computation

Problem 2.[15 points] Select the correct answer for each of the following questions. (Each question has exactly one correct answer).

(1) The general solution of the differential equation $y'' + 5y' = 0$ is

(a) $y = c_1 \cos 5x + c_2 \sin 5x$
(b) $y = c_1 + c_2 e^{-5x}$
(c) $y = c_1 + c_2 e^{5x}$
(d) $y = c_1 e^{-5x} + c_2 x e^{-5x}$

Answer: B

(2) An equation with general solution $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$ is

(a) $y'' - 2y' + 5 = 0$

(b) $y'' - 2y' - 5y = 0$
(c) $y'' - 2y' + 5y = 0$
(d) $y'' + 2y' + 5 = 0$

Answer: C

(3) An object of mass m is attached to a spring with spring constant k and the object undergoes free undamped motion. The period T of this motion is

(a) $T = 2\pi\sqrt{\frac{k}{m}}$
(b) $T = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
(c) $T = 2\pi\sqrt{\frac{m}{k}}$
(d) $T = 2\pi\sqrt{k^2 + m^2}$

Answer: C

Problem 3.[20 points] Solve the initial value problem

$$y'' - 2y' - 3y = 0, \quad y(0) = 6, y'(0) = -2.$$

Give all the details of your work.
bf Solution.

The characteristic equation is $r^2 - 2r - 3 = 0, (r-3)(r+1) = 0$ so the roots are $r_1 = 3, r_2 = -1$. Therefore the general solution of our differential equation is $y = Ae^{3x} + Be^{-x}$. Thus $y' = 3Ae^{3x} - Be^{-x}$. Substituting the intial conditions $y(0) = 6, y'(0) = -2$ we get

$$\begin{cases} A + B = 6, \\ 3A - B = -2 \end{cases} \Rightarrow \begin{cases} A = 1, \\ B = 5 \end{cases}$$

Therefore $y = e^{3x} + 5e^{-x}$.

Problem 4.[20 points] Find a particular solution of the equation $y'' + y' - 2y = 6e^x$. Give all the details of your work.

The characteristic equation is

$$r^2 + r - 2 = 0, \quad (r+2)(r-1) = 0, \quad r_1 = -2, r_2 = 1$$

Therefore the general solution of the complimentary homogeneous problem is $y_c = c_1e^{-2x} + c_2e^x$. The function $6e^x$ is of the form (polynomial of degree zero) e^x. Hence we will look for a particular solution in the form $y_p = x^s$(polynomial of degree zero) $e^x = x^sAe^x = xAe^x = Axe^x$.

Thus

$$y = Axe^x, y' = Ae^x + Axe^x, y'' = 2Ae^x + Axe^x.$$

Subsitituting this data into the main equation we get

$$\begin{aligned} &(2Ae^x + Axe^x) + (Ae^x + Axe^x) - 2Axe^x = 6xe^x \\ &3Ae^x = 6e^x, \quad (3A - 6)e^x = 0, \quad 3A - 6 = 0 \\ &A = 2 \end{aligned}$$

Thus $y_p = 2xe^x$.

Problem 5.[30 points] A differential equation $y''+P(x)y'+Q(x)y=0$ has solutions $y_1=\sin^2 x$ and $y_2=\sin x$.

Find a particular solution of the equation

$$y''+P(x)y'+Q(x)y=\cot x$$

Give all the details of your work.

Solution

We will use the method of Variation of Parameters. Thus we can look for a particular solution in the form $y_p=u_1y_1+u_2y_2$ where

$$\begin{cases} u_1'y_1+u_2'y_2=0 \\ u_1'y_1'+u_2''y_2'=f(x). \end{cases}$$

Thus

$$\begin{cases} u_1'\sin^2 x+u_2'\sin x=0 \\ u_1'2\sin x\cos x+u_2'\cos x=\cot x \end{cases} \Rightarrow \begin{cases} u_2'=-(\sin x)u_1 \\ u_1'2\sin x\cos x-u_1'\sin x\cos x=\dfrac{\cos x}{\sin x} \end{cases} \Rightarrow$$

$$\begin{cases} u_2'=-(\sin x)u_1 \\ u_1'\sin x\cos x=\dfrac{\cos x}{\sin x} \end{cases} \Rightarrow \begin{cases} u_1'=\dfrac{1}{\sin^2 x}=\csc^2 x \\ u_2'=-(\sin x)u_1'=-\dfrac{1}{\sin x}=-\csc x \end{cases}$$

Hence $u_1=\int\csc^2 x\,dx=-\cot x$ and $u_2=\int-\csc x\,dx=-\ln|\csc x-\cot x|$. Therefore

$$y_p=u_1y_1+u_2y_2=-\cot x\sin^2 x-\sin x\ln|\csc x-\cot x|=-\cos x\sin x\sin x-\ln|\csc x-\cot x|.$$