

Quiz 3; October 27 (with Solutions)

(1) Set up the appropriate form of a particular solution y_p (but do not determine the values of the coefficients) for the following equation:

$$y'' + 6y' + 9y = xe^{-3x}.$$

Solution.

The associated homogeneous equation $y'' + 6y' + 9y = 0$ has characteristic equation $r^2 + 6r + 9 = (r + 3)^2 = 0$. Therefore the complimentary solution is $y_c = c_1e^{-3x} + c_2xe^{-3x}$. The function $f(x) = xe^{-3x}$ on the right-hand side of the original equation is of the form

$$(1\text{-st degree polynomial}) e^{-3x}.$$

Therefore we should look for y_p as $y_p = x^s(A + Bx)e^{-3x}$ where $s \geq 0$ is the smallest number eliminating duplication of the terms from y_c in y_p . In this case $s = 2$ and y_p has the form

$$y_p = x^2(A + Bx)e^{-3x} = Ax^2e^{-3x} + Bx^3e^{-3x}.$$

(2) For the following problem determine if $\lambda = 1/4$ is an eigenvalue and if yes, find the corresponding eigenfunction.

$$y'' + \lambda y = 0, \quad y'(-\pi) = y'(\pi) = 0.$$

Solution.

With $\lambda = 1/4$ our equation becomes $y'' + \frac{1}{4}y = 0$. Its characteristic equation is $r^2 + \frac{1}{4} = 0$ which has roots $r = \pm \frac{i}{2}$.

Therefore $y = A \cos \frac{x}{2} + B \sin \frac{x}{2}$. By differentiating we get

$$y' = -\frac{A}{2} \sin \frac{x}{2} + \frac{B}{2} \cos \frac{x}{2}.$$

Using $y'(\pi) = 0$, the above formula yields

$$-\frac{A}{2} \cdot 1 + \frac{B}{2} \cdot 0 = 0, \text{ and hence } A = 0.$$

Similarly, using $y'(-\pi) = 0$ we get

$$-\frac{A}{2} \cdot (-1) + \frac{B}{2} \cdot 0 = 0, \text{ and hence } A = 0.$$

Thus with $A = 0$ and arbitrary B the function $y = A \cos \frac{x}{2} + B \sin \frac{x}{2} = B \sin \frac{x}{2}$ solves the system

$$y'' + \frac{1}{4}y = 0, \quad y'(-\pi) = y'(\pi) = 0.$$

Therefore $\lambda = \frac{1}{4}$ is an eigenvalue and $y = \sin \frac{x}{2}$ is the corresponding eigenfunction.