Quiz 4, November 10, 2003 (with Solutions)

Problem 1. In this problem choose the correct option from the choices provided (no explanations are necessary).

Let $f(x)$ be a 2π-periodic function such that

$$f(x) = \begin{cases} x^7, & 0 < x < \pi \\ 3x - 2, & -\pi < x < 0 \\ 10, & x = 0, \pi, -\pi. \end{cases}$$

Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series of $f(x)$.

Then:

(1) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = 0$.
(2) the series $\sum_{n=1}^{\infty} a_n$ diverges.
(3) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = 5$.
(4) None of the above.

Answer: (4)

Solution.

Since $\sin 0 = 0$ and $\cos 0 = 1$, the expression $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$ is precisely the Fourier series of $f(x)$ with $x = 0$. The function $f(x)$ is piecewise-continuous and at $x = 0$ it has a discontinuity with $f(0-) = 3 \cdot 0 - 2 = -2$ and $f(0+) = 0^7 = 0$. Therefore by Theorem 1 on page 604 the Fourier series of $f(x)$ at $x = 0$ converges to $\frac{1}{2}(f(0-) + f(0+)) = \frac{1}{2}(-2 + 0) = -1$.

Thus $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = -1$.

Therefore the correct choice is (4).

Problem 2.

Let $f(x)$ be a 4-periodic function such that

$$f(x) = \begin{cases} 5, & 0 < x < 2 \\ -5, & -2 < x < 0 \\ 0, & x = 0, 2, -2. \end{cases}$$

Compute the coefficient b_3 of the general Fourier series of $f(x)$. Give the details of your work.

Solution.

The half-period L equals $L = 4/2 = 2$. Therefore

$$b_3 = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{3\pi x}{2}\, dx = \frac{2}{2}\int_{0}^{2} f(x) \sin\frac{3\pi x}{2}\, dx,$$

where the second equality holds since the function $f(x)$ is odd and hence the function $f(x)\sin\frac{3\pi x}{2}$ is even.

Thus

$$b_3 = \int_0^2 f(x) \sin \frac{3\pi x}{2}\, dx = 5 \int_0^2 \sin \frac{3\pi x}{2}\, dx = -5 \cdot \frac{2}{3\pi} [\cos \frac{3\pi x}{2}]_0^2 =$$
$$= \frac{-10}{3\pi} [\cos 3\pi\ - 1] = \frac{-10}{3\pi} [(-1) - 1] = \frac{20}{3\pi}$$