Some Review Problems (with Solutions)

Problem 1. Show that the functions

$$f(x) = \begin{cases} 1 & x \text{ rational}, \\ -1, & x \text{ irrational} \end{cases}$$

and $g(x) \equiv 3$ are linearly independent on the interval $(-\infty, \infty)$.

Solution.

Suppose, on the contrary, that $f(x)$ and $g(x)$ are linearly dependent on $(-\infty, \infty)$. Since neither f nor g is identically equal to zero, this implies that $f \equiv cg$ for some constant $c \neq 0$.

For $x = 0$ the equation $f(0) = cg(0)$ gives us $1 = 3c$, so that $c = 1/3$. On the other hand for $x = \pi$ (an irrational number) we have $f(\pi) = cg(\pi)$, so that $-1 = 3c$ and $c = -1/3$. Since $1/3 \neq -1/3$, we obtain a contradiction.

Thus the assumption that $f(x)$ and $g(x)$ are linearly dependent on $(-\infty, \infty)$ was false and hence $f(x)$ and $g(x)$ are linearly independent on $(-\infty, \infty)$.

Problem 2.

Find all the separated solutions of the system:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi, \\ u(0, y) = u(\pi, y) = 0, & 0 < y < \pi \\ u(x, \pi) = 0, & 0 < x < \pi. \end{cases}$$

Note: There was a misprint in the statement of this problem in the class handout

Solution.

We need to look for separated solutions of the form $u = X(x)Y(y)$, where $X(x), Y(y)$ are some functions that are not identically equal to zero.

Substituting $u = X(x)Y(y)$ in $u_{xx} + u_{yy} = 0$ we get $X''Y + Y''X = 0$ and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = const.$$

This gives us $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. The conditions $u(0, y) = u(\pi, y) = 0$ imply $X(0) = X(\pi) = 0$. Thus we get an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $X_n = \sin nx$, $n = 1, 2, 3, \ldots$.

For each $\lambda_n = n^2$ we have the equation $Y'' - \lambda_n Y = 0$ which gives $Y'' - n^2 Y = 0$. The condition $u(x, \pi) = 0$ gives us $X(x)Y(\pi) = 0$, $Y(\pi) = 0$. The general solution of $Y'' - n^2 Y = 0$ is $Y = A \cosh ny + B \sinh ny$. The condition $Y(\pi) = 0$ gives us $A \cosh \pi n + B \sinh \pi n = 0$, so that $B = -A\frac{\cosh \pi n}{\sinh \pi n}$. Now $Y = A \cosh ny + B \sinh ny$ takes the form

$$\begin{aligned}Y =&A\cosh ny + B\sinh ny = A\cosh ny - A\frac{\cosh \pi n}{\sinh \pi n}\sinh ny = \\ &\frac{A}{\sinh \pi n}(\sinh \pi n \cosh ny - \cosh \pi n \sinh ny) = \\ &\frac{A}{\sinh \pi n}\sinh n(\pi - y)\end{aligned}$$

For the last equality we used the identity $\sinh(a-b) = sinha \cosh b - \cosh a \sinh b$ with $a = \pi n$ and $b = ny$. Since $\frac{A}{\sinh \pi n}$ is a constant not depending on y, we can take $Y = Y_n = \sinh n(\pi - y)$.

Thus the separated solutions are:

$$u_n(x,t) = X_nY_n = \sin nx \, \sinh n(\pi - y), \quad n = 1, 2, 3, \dots.$$

Problem 3.

Solve the system:

$$\begin{cases} y_{tt} = 25y_{xx}, \quad 0 < x < 3, t > 0, \\ y(0,t) = y(3,t) = 0, \quad t > 0 \\ y(x,0) = 0, \quad 0 < x < 3 \\ y_t(x,0) = 10\sin 2\pi x, \quad 0 < x < 3. \end{cases}$$

Solution.

This is Problem B (for the wave equation) with $L = 3$ and $a = \sqrt{25} = 5$ and $g(x) = 10\sin 2\pi x$. Therefore it has solution of the form

$$y = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n a t}{L}\sin\frac{\pi n x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{5\pi n t}{3}\sin\frac{\pi n x}{3},$$

where $B_n = \frac{L}{\pi n a}b_n = \frac{3}{5\pi n}b_n$ and $g(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{3}$ is the Fourier Sine Series of $g(x)$.

We have

$$g(x) = 10\sin 2\pi = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{3}, \quad 0 < x < 3$$

and hence $b_6 = 10$ and $b_n = 0$ for $n \neq 6$. Thus $B_6 = \frac{3}{5\pi 6}b_6 = \frac{30}{30\pi} = \frac{1}{\pi}$ and $B_n = 0, n \neq 6$. Hence

$$y = \frac{1}{\pi}\sin 10\pi t \, \sin 2\pi x.$$

Problem 4.

An object of mass $m = 1kg$ is attached to a spring with Hooke's constant $k = 4N/m$ and is acted on by a 2π-periodic force $F(t)$ Newtons where $F(t) = 1$ for $0 < t < \pi$ and $F(t) = -1$ for $-\pi < t < 0$.

Determine whether or not pure resonance occurs.

Solution.

The dispacement function $x(t)$ satisfies the equation

$$mx'' + kx = F(t), \quad x'' + \frac{k}{m} = \frac{F(t)}{m}, \quad x'' + 4x = F(t)$$

The natural frequency of the system is $\omega_0 = \sqrt{\frac{k}{m}} = 2$. Let $F(t) \sim \sum_{n=1}^{\infty} b_n \sin \frac{\pi n t}{L} = \sum_{n=1}^{\infty} b_n \sin nt$ be the Fourier Series of the odd function $F(t)$. Pure resonance occurs if there is n such that $\omega_0 = \frac{\pi n}{L}$ and $b_n \neq 0$.

The condition $\omega_0 = 2 = \frac{\pi n}{L} = n$ holds for $n = 2$.

We have

$$\begin{aligned} b_2 =& \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin 2t \, dt = \qquad \text{since } F(t) \text{ is odd} \\ =& \frac{2}{\pi} \int_{0}^{\pi} F(t) \sin 2t \, dt = \frac{2}{\pi} \int_{0}^{\pi} \sin 2t \, dt = \\ =& \frac{2}{\pi} \frac{-1}{2} [\cos 2t]_0^{\pi} = 0. \end{aligned}$$

Since $b_2 = 0$, pure resonance does not occur.

Problem 5.

Let $f(x)$ be a 6-periodic function such that $f(x) = x^2 - x$ for $-3 < x < 3$. Let a_n, b_n be the general Fourier series coefficients of $f(x)$. Find

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{46\pi n}{3} + b_n \sin \frac{46\pi n}{3})$$

and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \pi n + b_n \sin \pi n).$$

Answers:

We have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{46\pi n}{3} + b_n \sin \frac{46\pi n}{3}) = 6$$

and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \pi n + b_n \sin \pi n) = 9.$$

Problem 6.

Find the Fourier Sine Series of the function $f(x)$ defined on the interval $0 < x < 3$ as

$$f(x) = \begin{cases} 1 & 0 < x \leq 2, \\ 5, & 2 < x < 3. \end{cases}$$

Problem 7.

Find the general solution of the following equation:

$$y'' - 6y' + 9y = x^2 e^{3x} + \cos x.$$

Note: This is a pretty long problem (and I did make an arithmetic mistake in class when computing y_2).

Solution.

The general solution has the form $y = y_c + y_1 + y_2$ where y_c is the general solution of the complimentary homogeneous problem $y'' - 6y' + 9y = 0$, where y_1 is a particular solution of $y'' - 6y' + 9y = x^2e^{3x}$ and where y_2 is a particular solution of $y'' - 6y' + 9y = \cos x$.

The equation $y''-6y'+9y = 0$ has characteristic equation $r^2-6r+9 = (r-3)^2 = 0$ and so $y_c = c_1e^{3x} + c_2xe^{3x}$.

The Method of Undetermined coefficients tells us that we should look for y_1 in the form

$$y_1 = x^2(A + Bx + Cx^2)e^{3x} = (Ax^2 + Bx^3 + Cx^4)e^{3x}.$$

Therefore

$$\begin{aligned} y_1' &= (2Ax + 3Bx^2 + 4Cx^3)e^{3x} + (3Ax^2 + 3Bx^3 + 3Cx^4)e^{3x} = \\ &= (2Ax + (3A + 3B)x^2 + (3B + 4C)x^3 + 3Cx^4)e^{3x} \end{aligned}$$

and

$$\begin{aligned} y_1'' &= [2A + (6A + 6B)x + (9B + 12C)x^2 + 12Cx^3)e^{3x} + (6Ax + (9A + 9B)x^2 + \\ &+ (9B + 12C)x^3 + 9Cx^4]e^{3x} = \\ &= [2A + (12A + 6B)x + (9A + 9B + 12C)x^2 + (9B + 24C)x^3 + 9Cx^4]e^{3x}. \end{aligned}$$

Substituting this data in $y'' - 6y' + 9y = x^2e^{3x}$ and re-groupping we get

$$\begin{aligned} &[2A + (6A + 6B - 12A)x + (9A + 9B + 12C - 6(3A + 3B) + 9A)x^2 + \\ &(9B + 24C - 6(3B + 4C) + 9B)x^3 + (9C - 18C + 9C)x^4]e^{3x} = x^2e^{3x} \end{aligned}$$

Hence $2A = 0$, $-6A + 6B = 0$, $-9B + 12C = 1$, so that $A = B = 0, C = 1/3$. Thus $y_1 = \frac{1}{3}x^4e^{3x}$.

By the Method of Undetermined coefficients we should look for y_2 in the form $y_2 = E\cos x + F\sin x$. Then $y_2' = -E\sin x + F\cos x$ and $y_2'' = -E\cos x - F\sin x$. Substituting this information in $y'' - 6y' + 9y = \cos x$ we get

$$(-E\cos x - F\sin x) - 6(-E\sin x + F\cos x) + 9(E\cos x + F\sin x) = \cos x,$$

and

$$(-E - 6F + 9E)\cos x + (-F + 6E + 9F)\sin x = \cos x.$$

Hence $8E - 6F = 1$, $6E + 8F = 0$ and therefore $E = 2/25$, $F = -3/50$ and $y_2 = \frac{2}{25}\cos x - \frac{3}{50}\sin x$.

Thus the general solution of the equation $y'' - 6y' + 9y = x^2e^{3x} + \cos x$ is

$$y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{3}x^4e^{3x} + \frac{2}{25}\cos x - \frac{3}{50}\sin x,$$

where c_1, c_2 are arbitrary constants.

Problem 8.

Let $y(x,t)$ be the solution of the system:

$$\begin{cases} y_{tt} = 4y_{xx}, \quad 0 < x < 1, t > 0, \\ y(0,t) = y(1,t) = 0, \quad t > 0 \\ y(x,0) = x^2, \quad 0 < x < 1 \\ y_t(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

Using d'Alambert's method find the precise value of $y(1/2, 10)$.

Solution.

In this problem $a = \sqrt{4} = 2$, $L = 1$ and $f(x) = x^2$ for $0 < x < 1$.

According to d'Alambert's method we have $y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)]$ where $F(x) = f_O(x)$ is the $2L$-periodic odd extension of $f(x)$.

So

$$y(1/2, 10) = \frac{1}{2}[F(1/2 + 2 \cdot 10) + F(1/2 - 2 \cdot 10)] = \frac{1}{2}[F(20.5) + F(-19.5)].$$

The function $F(x)$ is 2-periodic and therefore

$$F(20.5) = F(1/2 + 2 \cdot 10) = F(1/2) = f(1/2) = (1/2)^2 = 1/4$$
$$F(-19.5) = F(1/2 - 2 \cdot 10) = F(1/2) = f(1/2) = (1/2)^2 = 1/4.$$

Hence

$$y(1/2, 10) = \frac{1}{2}[\frac{1}{4} + \frac{1}{4}] = \frac{1}{4}.$$