Math 285 Section C1 Exam 1 (Solutions)
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Problem 1. [16 points] Select the correct answer for each of the following questions. Each question has exactly one correct answer. **You do not need to provide any explanations in this problem.**

(1) For the initial value problem $\frac{dy}{dx} = \sin(x^2 + y^4)$, $y(1) = 4$ on the interval $I = (-\infty, \infty)$

(a) A solution is guaranteed to exist on I.
(b) A solution is guaranteed to exist on an interval $(1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$.
(c) A unique solution is guaranteed to exist on the interval I.
(d) There are infinitely many solutions on I.

Answer: B

(2) The differential equation $y'' - 2xy' + x^3y = 0$ on the interval $I = (-\infty, \infty)$

(a) has exactly one solution;
(b) has the property that for any solutions y_1, y_2 the function $y_1 + 5y_2$ is also a solution;
(c) has characteristic equation $r^2 - 2r + 1 = 0$;
(d) none of the above.

Answer B

(3) Making the substitution $v = y/x$ in an equation $y' = F(y/x)$ transforms this differential equation into

(a) a linear first order differential equation;
(b) a separable differential equation;
(c) an exact equation;
(d) a homogeneous equation;

Answer: B

(4) The differential equation $(2x + y^2)dx + (2y + x^2)dy = 0$ is

(a) linear;
(b) homogeneous;
(c) exact;
(d) none of the above.

Answer: D

Problem 2.[24 points] Find the general solution of the following differential equations. Find explicit expressions for y in terms of x, when possible.

Give all the details of your work.

(a) $y' = (x+y-3)^2$;

Solution.

Use the substitution $v = x+y-3$. Then $y = v - x + 3$, $\frac{dy}{dx} = \frac{dv}{dx} - 1$ and hence the equation becomes

$$\frac{dv}{dx} - 1 = v^2, \quad \frac{dv}{dx} = v^2 + 1, \quad \frac{dv}{v^2+1} = dx$$
$$\int \frac{dv}{v^2+1} = \int dx, \quad \arctan v = x + C \quad v = \tan(x+C)$$
$$x + y - 3 = \tan(x+C), \quad y = \tan(x+C) - x + 3.$$

(b) $y' = \frac{x-y}{x+y}$.

Solution.

We have

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-y/x}{1+y/x},$$

so that this equation is homogeneous. Use the substitution $v = y/x$, so that $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}, \quad x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$$
$$\frac{1+v}{1-2v-v^2}\,dv = \frac{1}{x}\,dx$$
$$\int \frac{1+v}{1-2v-v^2}\,dv = \frac{1}{x}\,dx = \ln|x| + C.$$

We have

$$\int \frac{1+v}{1-2v-v^2}\,dv = \frac{1}{2}\int \frac{d(v^2+2v)}{1-2v-v^2} = -\frac{1}{2}\ln|1-2v-v^2| = \ln|x| + C$$
$$\ln|1-2v-v^2| = -2\ln|x| - 2C, \quad |1-2v-v^2| = e^{-2C}|x|^{-2} = e^{-2C}\frac{1}{x^2}$$
$$1 - 2v - v^2 = \pm e^{-2C}\frac{1}{x^2} = \frac{A}{x^2}, \text{ where } A \text{ is a constant}$$
$$1 - 2\frac{y}{x} - \frac{y^2}{v^2} = \frac{A}{x^2}$$
$$x^2 - 2xy - y^2 = A.$$

Thus $x^2 - 2xy - y^2 = A$, where A is a constant, is the general soultion of the original equation defining $y = y(x)$ implicitly.

Problem 3.[18 points] Find the general solution of the equation

$$y' + \frac{3y}{x} = \sqrt{x}$$

on the interval $x > 0$. Give all the details of your work.
Solution.

This is a linear equation of order 1 with $P(x) = \frac{3}{x}$ and $Q(x) = \sqrt{x}$. We first compute the integrating factor:

$$\rho(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3.$$

Multiplying both sides of the original equation by x^3 we get

$$x^3\frac{dy}{dx} + 3yx^2 = x^{7/2}$$
$$\frac{d}{dx}(x^3y) = x^{7/2}$$
$$x^3y = \int x^{7/2} = \frac{2}{9}x^{9/2} + C$$
$$y = \frac{2}{9}x^{3/2} + \frac{C}{x^3},$$

where C is a constant.

Problem 4.[22 points] Find the general solution of the following differential equations. Find explicit expressions for y in terms of x, when possible. Give all the details of your work.

(a) $(2y + 1)dx + (2x + 3y^2 - 4)dy = 0$

Solution.

Since $\frac{d}{dy}(2y + 1) = 2 = \frac{d}{dx}(2x + 3y^2 - 4)$, the equation is exact.

We need to find $F(x, y)$ such that $\frac{\partial F}{\partial x} = 2y+1$ and $\frac{\partial F}{\partial y} = 2x+3y^2-4$. We get

$$F(x, y) = \int (2y + 1)\,dx = 2yx + x + g(y).$$

Now the condition $\frac{\partial F}{\partial y} = 2x + 3y^2 - 4$ gives us

$$2x + g'(y) = 2x + 3y^2 - 4$$
$$g'(y) = 3y^2 - 4, \quad g(y) = \int (3y^2 - 4)\,dy = y^3 - 4y$$

Thus $F(x, y) = 2yx + x + y^3 - 4y$ satisfies our requirements and the general solution of the original equation is implicitly given by

$$2yx + x + y^3 - 4y = C,$$

where C is a constant.

(b) $y'' - 2y' - 3y = 0$.

Solution.

This is a homogeneous linear equation of order two with constant coefficients. Its characteristic equation is $r^2-2r-3=0$, $(r-3)(r+1)=0$. Thus the roots are $r_1 = 3$, $r_2 = -1$. Hence the general solution is

$$y = c_1e^{3x} + c_2e^{-x},$$

where c_1, c_2 are arbitrary constants.

Problem 5. [20 points]

Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are linearly independent on the interval $(-\infty, \infty)$ but are linearly dependent on the interval $[1, \infty)$ and also are linearly dependent on the interval $(-\infty, 1]$. **Carefully justify** that your functions have the required properties.

Solution.

Let $f(x) = x - 1$, $g(x) = |x - 1|$. Thus $g(x) = x - 1$ for $x \geq 1$ and $g(x) = 1 - x$ for $x \leq 1$.

We see that on the interval $[1, \infty)$ we have $f(x) = g(x) = g(x)$ and so the functions f and g are linearly dependent on that interval.

Similarly, on the interval $(-\infty, 1]$ we have $f(x) = -g(x)$ and hence f, g are linearly dependent on that interval as well.

Suppose now that f and g are linearly dependent on $(-\infty, \infty)$. Since both f and g are not identically zero, this means that there is some $c \neq 0$ such that $f(x) = cg(x)$ for every $x \in (-\infty, \infty)$. For $x = 2$ we have $1 = f(2) = cg(2) = c \cdot 1 = c$, so that $c = 1$. Thus $f = 1 \cdot g = g$ on $(-\infty, \infty)$. However, for $x = 0$ we have

$$f(0) = 0 - 1 = -1 \neq 1 = |0 - 1| = g(0).$$

This gives us a contradiction. Thus the assumption that f and g are linearly dependent on $(-\infty, \infty)$ was false, and hence f and g are linearly independent on on $(-\infty, \infty)$, as required.

Here is another example of (discontinuous) functions f and g with the required properties:

$$f(x) = \begin{cases} 1, & \text{if } x \neq 1, \\ 0, & \text{if } x = 1, \end{cases}$$

$$g(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \\ 2. & \text{if } x > 1. \end{cases}$$