

Some Review Problems

Problem 1. Show that the functions

$$f(x) = \begin{cases} 1 & x \text{ rational,} \\ -1, & x \text{ irrational} \end{cases}$$

and $g(x) \equiv 3$ are linearly independent on the interval $(-\infty, \infty)$.

Solution.

Suppose, on the contrary, that $f(x)$ and $g(x)$ are linearly dependent on $(-\infty, \infty)$. Since neither f nor g is identically equal to zero, this implies that $f \equiv cg$ for some constant $c \neq 0$.

For $x = 0$ the equation $f(0) = cg(0)$ gives us $1 = 3c$, so that $c = 1/3$. On the other hand for $x = \pi$ (an irrational number) we have $f(\pi) = cg(\pi)$, so that $-1 = 3c$ and $c = -1/3$. Since $1/3 \neq -1/3$, we obtain a contradiction.

Thus the assumption that $f(x)$ and $g(x)$ are linearly dependent on $(-\infty, \infty)$ was false and hence $f(x)$ and $g(x)$ are linearly independent on $(-\infty, \infty)$.

Problem 2.

Find all the separated solutions of the system:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi, \\ u(0, y) = u(\pi, y) = 0, & 0 < y < \pi \\ u(x, \pi) = 0, & 0 < x < \pi. \end{cases}$$

Solution.

We need to look for separated solutions of the form $u = X(x)Y(y)$, where $X(x), Y(y)$ are some functions that are not identically equal to zero.

Substituting $u = X(x)Y(y)$ in $u_{xx} + u_{yy} = 0$ we get $X''Y + Y''X = 0$ and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{const.}$$

This gives us $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. The conditions $u(0, y) = u(\pi, y) = 0$ imply $X(0) = X(\pi) = 0$. Thus we get an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = n^2$ for $n = 1, 2, 3, \dots$ and the corresponding eigenfunctions are $X_n = \sin nx$, $n = 1, 2, 3, \dots$.

For each $\lambda_n = n^2$ we have the equation $Y'' - \lambda_n Y = 0$ which gives $Y'' - n^2 Y = 0$. The condition $u(x, \pi) = 0$ gives us $X(x)Y(\pi) = 0$, $Y(\pi) = 0$. The general solution of $Y'' - n^2 Y = 0$ is $Y = A \cosh ny + B \sinh ny$. The condition $Y(\pi) = 0$ gives us $A \cosh \pi n + B \sinh \pi n = 0$, so that $B = -A \frac{\cosh \pi n}{\sinh \pi n}$. Now $Y = A \cosh ny + B \sinh ny$ takes the form

$$\begin{aligned} Y &= A \cosh ny + B \sinh ny = A \cosh ny - A \frac{\cosh \pi n}{\sinh \pi n} \sinh ny = \\ &= \frac{A}{\sinh \pi n} (\sinh \pi n \cosh ny - \cosh \pi n \sinh ny) = \\ &= \frac{A}{\sinh \pi n} \sinh n(\pi - y) \end{aligned}$$

For the last equality we used the identity

$$\sinh(a - b) = \sinh a \cosh b - \cosh a \sinh b$$

with $a = \pi n$ and $b = ny$. Since $\frac{A}{\sinh \pi n}$ is a constant not depending on y , we can take $Y = Y_n = \sinh n(\pi - y)$.

Thus the separated solutions are:

$$u_n(x, y) = X_n Y_n = \sin nx \sinh n(\pi - y), \quad n = 1, 2, 3, \dots$$

Problem 3.

Solve the system:

$$\begin{cases} y_{tt} = 25y_{xx}, & 0 < x < 3, t > 0, \\ y(0, t) = y(3, t) = 0, & t > 0 \\ y(x, 0) = 0, & 0 < x < 3 \\ y_t(x, 0) = 10 \sin 2\pi x, & 0 < x < 3. \end{cases}$$

This is Problem B (for the wave equation) with $L = 3$ and $a = \sqrt{25} = 5$ and $g(x) = 10 \sin 2\pi x$. Therefore it has solution of the form

$$y = \sum_{n=1}^{\infty} B_n \sin \frac{\pi n a t}{L} \sin \frac{\pi n x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{5\pi n t}{3} \sin \frac{\pi n x}{3},$$

where $B_n = \frac{L}{\pi n a} b_n = \frac{3}{5\pi n} b_n$ and $g(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{3}$ is the Fourier Sine Series of $g(x)$.

We have

$$g(x) = 10 \sin 2\pi = \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{3}, \quad 0 < x < 3$$

and hence $b_6 = 10$ and $b_n = 0$ for $n \neq 6$. Thus $B_6 = \frac{3}{5\pi 6} b_6 = \frac{30}{30\pi} = \frac{1}{\pi}$ and $B_n = 0, n \neq 6$. Hence

$$y = \frac{1}{\pi} \sin 10\pi t \sin 2\pi x.$$

Problem 4.

An object of mass $m = 1\text{kg}$ is attached to a spring with Hooke's constant $k = 4N/m$ and is acted on by a 2π -periodic force $F(t)$ Newtons where $F(t) = 1$ for $0 < t < \pi$ and $F(t) = -1$ for $-\pi < t < 0$.

Determine whether or not pure resonance occurs.

Solution.

The displacement function $x(t)$ satisfies the equation

$$mx'' + kx = F(t), \quad x'' + \frac{k}{m} = \frac{F(t)}{m}, \quad x'' + 4x = F(t)$$

The natural frequency of the system is $\omega_0 = \sqrt{\frac{k}{m}} = 2$. Let $F(t) \sim \sum_{n=1}^{\infty} b_n \sin \frac{\pi n t}{L} = \sum_{n=1}^{\infty} b_n \sin nt$ be the Fourier Series of the odd function $F(t)$. Pure resonance occurs if there is n such that $\omega_0 = \frac{\pi n}{L}$ and $b_n \neq 0$.

The condition $\omega_0 = 2 = \frac{\pi n}{L} = n$ holds for $n = 2$.

We have

$$\begin{aligned} b_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin 2t \, dt = \quad \text{since } F(t) \text{ is odd} \\ &= \frac{2}{\pi} \int_0^{\pi} F(t) \sin 2t \, dt = \frac{2}{\pi} \int_0^{\pi} \sin 2t \, dt = \\ &= \frac{2}{\pi} \frac{-1}{2} [\cos 2t]_0^{\pi} = 0. \end{aligned}$$

Since $b_2 = 0$, pure resonance does not occur.

Problem 5.

Let $f(x)$ be a 6-periodic function such that $f(x) = x^2 - x$ for $-3 < x < 3$. Let a_n, b_n be the general Fourier series coefficients of $f(x)$. Find

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{46\pi n}{3} + b_n \sin \frac{46\pi n}{3} \right)$$

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and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \pi n + b_n \sin \pi n).$$

Answers:

We have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{46\pi n}{3} + b_n \sin \frac{46\pi n}{3} \right) = 6$$

and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \pi n + b_n \sin \pi n) = 9.$$

Problem 6.

Find the Fourier Sine Series of the function $f(x)$ defined on the interval $0 < x < 3$ as

$$f(x) = \begin{cases} 1 & 0 < x \leq 2, \\ 5, & 2 < x < 3. \end{cases}$$

Problem 7.

Find the general solution of the following equation:

$$y'' - 6y' + 9y = x^2 e^{3x} + \cos x.$$

Note: This is a pretty long problem!

Solution.

The general solution has the form $y = y_c + y_1 + y_2$ where y_c is the general solution of the complimentary homogeneous problem $y'' - 6y' + 9y = 0$, where y_1 is a particular solution of $y'' - 6y' + 9y = x^2 e^{3x}$ and where y_2 is a particular solution of $y'' - 6y' + 9y = \cos x$.

The equation $y'' - 6y' + 9y = 0$ has characteristic equation $r^2 - 6r + 9 = (r - 3)^2 = 0$ and so $y_c = c_1 e^{3x} + c_2 x e^{3x}$.

The Method of Undetermined coefficients tells us that we should look for y_1 in the form

$$y_1 = x^2(A + Bx + Cx^2)e^{3x} = (Ax^2 + Bx^3 + Cx^4)e^{3x}.$$

Therefore

$$\begin{aligned} y_1' &= (2Ax + 3Bx^2 + 4Cx^3)e^{3x} + (3Ax^2 + 3Bx^3 + 3Cx^4)e^{3x} = \\ &= (2Ax + (3A + 3B)x^2 + (3B + 4C)x^3 + 3Cx^4)e^{3x} \end{aligned}$$

and

$$\begin{aligned} y_1'' &= [2A + (6A + 6B)x + (9B + 12C)x^2 + 12Cx^3]e^{3x} + (6Ax + (9A + 9B)x^2 + \\ &+ (9B + 12C)x^3 + 9Cx^4]e^{3x} = \\ &= [2A + (12A + 6B)x + (9A + 9B + 12C)x^2 + (9B + 24C)x^3 + 9Cx^4]e^{3x}. \end{aligned}$$

Substituting this data in $y'' - 6y' + 9y = x^2e^{3x}$ and re-grouping we get

$$[2A + (6A + 6B - 12A)x + (9A + 9B + 12C - 6(3A + 3B) + 9A)x^2 + (9B + 24C - 6(3B + 4C) + 9B)x^3 + (9C - 18C + 9C)x^4]e^{3x} = x^2e^{3x}$$

Hence $2A = 0$, $-6A + 6B = 0$, $-9B + 12C = 1$, so that $A = B = 0$, $C = 1/3$. Thus $y_1 = \frac{1}{3}x^4e^{3x}$.

By the Method of Undetermined coefficients we should look for y_2 in the form $y_2 = E \cos x + F \sin x$. Then $y_2' = -E \sin x + F \cos x$ and $y_2'' = -E \cos x - F \sin x$. Substituting this information in $y'' - 6y' + 9y = \cos x$ we get

$$(-E \cos x - F \sin x) - 6(-E \sin x + F \cos x) + 9(E \cos x + F \sin x) = \cos x,$$

and

$$(-E - 6F + 9E) \cos x + (-F + 6E + 9F) \sin x = \cos x.$$

Hence $8E - 6F = 1$, $6E + 8F = 0$ and therefore $E = 2/25$, $F = -3/50$ and $y_2 = \frac{2}{25} \cos x - \frac{3}{50} \sin x$.

Thus the general solution of the equation $y'' - 6y' + 9y = x^2e^{3x} + \cos x$ is

$$y = c_1e^{3x} + c_2xe^{3x} + \frac{1}{3}x^4e^{3x} + \frac{2}{25} \cos x - \frac{3}{50} \sin x,$$

where c_1, c_2 are arbitrary constants.

Problem 8.

Let $y(x, t)$ be the solution of the system:

$$\begin{cases} y_{tt} = 4y_{xx}, & 0 < x < 1, t > 0, \\ y(0, t) = y(1, t) = 0, & t > 0 \\ y(x, 0) = x^2, & 0 < x < 1 \\ y_t(x, 0) = 0, & 0 < x < 1. \end{cases}$$

Using d'Alambert's solution find the precise value of $y(1/2, 10)$. Then find the formal infinite series solution of this system.

Solution.

In this problem $a = \sqrt{4} = 2$, $L = 1$ and $f(x) = x^2$ for $0 < x < 1$.

According to d'Alambert's method we have $y(x, t) = \frac{1}{2}[F(x + at) + F(x - at)]$ where $F(x) = f_O(x)$ is the $2L$ -periodic odd extension of $f(x)$.

So

$$y(1/2, 10) = \frac{1}{2}[F(1/2 + 2 \cdot 10) + F(1/2 - 2 \cdot 10)] = \frac{1}{2}[F(20.5) + F(-19.5)].$$

The function $F(x)$ is 2-periodic and therefore

$$F(20.5) = F(1/2 + 2 \cdot 10) = F(1/2) = f(1/2) = (1/2)^2 = 1/4$$

$$F(-19.5) = F(1/2 - 2 \cdot 10) = F(1/2) = f(1/2) = (1/2)^2 = 1/4.$$

Hence

$$y(1/2, 10) = \frac{1}{2}\left[\frac{1}{4} + \frac{1}{4}\right] = \frac{1}{4}.$$

Problem 9(1).

Solve the following Dirichlet Problem:

$$(*) \quad \begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b \\ u(x, 0) = u(x, b) = u(0, y) = 0, & 0 < x < a, 0 < y < b \\ u(a, y) = g(y), & 0 < y < b \end{cases}$$

Solution.

1) We first need to look for separated solutions of the form $u = X(x)Y(y)$, satisfying the first two equations of (*), where $X(x), Y(y)$ are some functions that are not identically equal to zero.

Substituting $u = X(x)Y(y)$ in $u_{xx} + u_{yy} = 0$ we get $X''Y + Y''X = 0$ and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = \text{const.}$$

This gives us $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$. The conditions $u(x, 0) = u(x, b) = 0$ imply $Y(0) = Y(b) = 0$. Thus we get an eigenvalue problem

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(b) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = \pi^2 n^2 / b^2$ for $n = 1, 2, 3, \dots$ and the corresponding eigenfunctions are $Y_n = \sin \frac{\pi n y}{b}$, $n = 1, 2, 3, \dots$

For each $\lambda_n = \pi^2 n^2 / b^2$ we have the equation $X'' - \lambda_n X = 0$ which gives $X'' - \pi^2 n^2 / b^2 X = 0$. The condition $u(0, y) = 0$ gives us $X(0)Y(y) = 0$, so that $X(0) = 0$. The general solution of $X'' - \pi^2 n^2 / b^2 X = 0$ is $X = A \cosh \frac{\pi n x}{b} + B \sinh \pi n x b$. The condition $X(0) = 0$ gives us $A \cdot 1 + B \cdot 0 = 0$, so that $A = 0$. Thus $X = B \sinh \frac{\pi n x}{b}$ and we can take $B = 1$ giving $X_n = \sinh \frac{\pi n x}{b}$.

Thus the separated solutions are:

$$u_n(x, y) = X_n Y_n = \sinh \frac{\pi n x}{b} \sin \frac{\pi n y}{b}, \quad n = 1, 2, 3, \dots$$

2) We now look for the formal solution $u(x, y)$ of the Dirichlet problem (*) in the form

$$(\dagger) \quad u = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \sinh \frac{\pi n x}{b} \sin \frac{\pi n y}{b}.$$

The condition $u(a, y) = g(y), 0 < y < b$ yields:

$$\sum_{n=1}^{\infty} c_n \sinh \frac{\pi n a}{b} \sin \frac{\pi n y}{b} = g(y), \quad 0 < y < b.$$

From here we conclude that $c_n \sinh \frac{\pi n a}{b} = b_n$, where b_n is the Fourier Sine Series coefficient for $g(y), 0 < y < b$.

Therefore the solution of the Dirichlet problem (*) is given by (\dagger) where

$$c_n = \frac{2}{b \sinh \frac{\pi n a}{b}} \int_0^b g(y) \sin \frac{\pi n y}{b} dy, \quad n = 1, 2, 3, \dots$$

Problem 9(2).

Solve the following Dirichlet Problem:

$$(**) \quad \begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, y > 0 \\ u(0, y) = u(a, y) = 0, & 0 < x < a, 0 < y < b \\ u(x, y) \text{ bounded as } y \rightarrow \infty, \\ u(x, 0) = f(x), & 0 < x < a \end{cases}$$

Solution.

1) We first need to look for separated solutions of the form $u = X(x)Y(y)$, satisfying the first three equations of (**), where $X(x), Y(y)$ are some functions that are not identically equal to zero.

Substituting $u = X(x)Y(y)$ in $u_{xx} + u_{yy} = 0$ we get $X''Y + Y''X = 0$ and hence

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{const.}$$

This gives us $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. The conditions $u(0, y) = u(a, y) = 0$ imply $X(0) = X(a) = 0$. Thus we get an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0. \end{cases}$$

The eigenvalues of this problem are $\lambda_n = \pi^2 n^2 / a^2$ for $n = 1, 2, 3, \dots$ and the corresponding eigenfunctions are $Y_n = \sin \frac{\pi n x}{a}$, $n = 1, 2, 3, \dots$

For each $\lambda_n = \pi^2 n^2 / a^2$ we have the equation $Y'' - \lambda_n Y = 0$ which gives $Y'' - \pi^2 n^2 / a^2 Y = 0$. The general solution of this equation is $Y = A \exp(\frac{\pi n y}{a}) + B \exp(-\frac{\pi n y}{a})$. The condition $u(x, y) = X(x)Y(y)$ bounded as $y \rightarrow \infty$ implies that $Y(y)$ is bounded as $y \rightarrow \infty$. This gives us $A = 0$. Thus $Y = B \exp(-\frac{\pi n y}{a})$. We can take the constant B to be $B = 1$ which yields $Y_n = \exp(-\frac{\pi n y}{a})$.

Thus the separated solutions are:

$$u_n(x, y) = X_n Y_n = \exp(-\frac{\pi n y}{a}) \sin \frac{\pi n x}{a}, \quad n = 1, 2, 3, \dots$$

2) We now look for the formal solution $u(x, y)$ of the Dirichlet problem (***) in the form

$$(\ddagger) \quad u = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \exp(-\frac{\pi n y}{a}) \sin \frac{\pi n x}{a}.$$

The condition $u(x, 0) = f(x)$, $0 < x < a$ yields:

$$\sum_{n=1}^{\infty} c_n \exp(-\frac{\pi n y}{a}) \sin \frac{\pi n x}{a} = f(x), \quad 0 < x < a.$$

From here we conclude that $c_n \exp(-\frac{\pi n 0}{a}) = c_n = b_n$, where b_n is the Fourier Sine Series coefficient for $f(x)$, $0 < x < a$.

Therefore the solution of the Dirichlet problem (*) is given by () where

$$c_n = \frac{2}{a} \int_0^a f(x) \sin \frac{\pi n x}{a} dx, \quad n = 1, 2, 3, \dots$$

Problem 10.

What does it mean for the functions $f(x), g(x)$ to be *orthogonal* on an interval $[a, b]$?

For each pair of the functions below determine if they are orthogonal on the interval indicated:

- (1) the functions $f(x) = \cos(3\pi x)$, $g(x) = \sin(4\pi x)$, on the interval $[-\pi/2, \pi/2]$;
- (2) the functions $f(x) = \cos(3\pi x)$, $g(x) = \sin(4\pi x)$, on the interval $[-4\pi, 4\pi]$;
- (3) the function $f(x) = \sin(x)$ and the function

$$g(x) = \begin{cases} \sin(x), & 0 \leq x \leq \pi, \\ 0, & -\pi \leq x < 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Problem 11 Consider the equation $y''' + 9y' = x \sin(3x)$.

Then according to the Method of Undetermined Coefficients a particular solution y_p can be found in the form:

- (a) $y_p = x(A \sin(3x) + B \cos(3x))$;
- (b) $y_p = x^2(A \sin(3x) + B \cos(3x))$;
- (c) $y_p = x^2(Ax + B) \sin(3x)$;
- (d) None of the above.

Answer: (d)

The correct form for y_p is

$$y_p = x[(Ax + B) \sin(3x) + (Cx + D) \cos(3x)].$$

Problem 12 Let $f(t) = t + 5$ for $0 < t < \pi$ and let b_n be the Fourier Sine Series Coefficients for $f(t)$. Then:

- (a) We have $1 = \sum_{n=1}^{\infty} nb_n \cos(nt)$ for each $t \in (-\infty, \infty)$.
- (b) We have $1 = \sum_{n=1}^{\infty} nb_n \cos(nt)$ for each $t \in (0, \pi)$.
- (c) We have $6 \neq \sum_{n=1}^{\infty} b_n \sin(n)$.
- (d) None of the above.

Answer: (d)

Note that term-wise differentiation is not applicable here since the odd extension $f_O(t)$ of $f(t)$ is not a continuous function on $(-\infty, \infty)$.

Problem 13.

Let $f(t) = 3t^2 - t^4$ for $0 < t < 2$. Let a_n , $n = 0, 1, 2, \dots$, be the coefficients of the Fourier Cosine Series of $f(t)$.

Find the precise values of the following expressions:

- (1) $\sum_{n=1}^{\infty} a_n$;
- (2) $\sum_{n=1}^{\infty} a_n \cos(\frac{5\pi n}{2})$;
- (3) $\sum_{n=1}^{\infty} \frac{\pi n}{2} a_n \sin(\frac{\pi n}{2})$.

[This was done in class on Wednesday, May 5]

Problem 14. Let $f(t)$ be a 6-periodic function defined as

$$f(t) = \begin{cases} t^2, & \text{if } 0 < t < 3, \\ 3t^2, & \text{if } 3 < t < 6, \\ 10, & \text{if } t = 0, 3, 6. \end{cases}$$

Let a_n, b_n be the general Fourier series coefficients for $f(t)$. Then:

- (a) We have $b_n = 0$ for all $n \geq 1$.
- (b) We have $a_n = 0$ for all $n \geq 0$.
- (c) We have $0 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$.
- (d) None of the above.

Answer: (d)

Note that for $-3 < t < 0$ we have $3 < t+6 < 6$ and $f(t) = f(t+6) = 3(t+6)^2$.

Problem 15.

Solve the following heat equation problem:

$$\begin{cases} y_t = 25y_{xx}, & 0 < x < 3, t > 0, \\ y(0, t) = y(3, t) = 0, & t > 0 \\ y(x, 0) = 4, & 0 < x < 3 \end{cases}$$

Solution.

We have $k = 25$, $L = 3$. By Theorem 1 in Ch. 9.5 the solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp\left(\frac{-n^2\pi^2 kt}{L^2}\right) \sin \frac{\pi nx}{L} = \sum_{n=1}^{\infty} b_n \exp\left(\frac{-25n^2\pi^2 t}{9}\right) \sin \frac{\pi nx}{3},$$

where b_n are the Fourier Sine Series Coefficients of the function $f(x) = 4, 0 < x < 3$.

We have

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 4 \sin \frac{\pi nx}{3} dx = \frac{8}{3} \frac{3}{\pi n} [-\cos \frac{\pi nx}{3}]_0^3 = \\ &= \frac{8}{\pi n} [-\cos(\pi n) + 1] = \frac{8}{\pi n} [-(-1)^n + 1] = \begin{cases} \frac{18}{\pi n}, & n \text{ odd,} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

Hence

$$u(x, t) = \sum_{n \text{ odd}} \frac{18}{\pi n} \exp\left(\frac{-25n^2\pi^2 t}{9}\right) \sin \frac{\pi nx}{3}$$

Problem 16

An object of mass $m = 2kg$ is attached to a spring with a spring constant $k = 18 N/m$ and is acted upon by an odd periodic force $F(t)$ Newtons.

In each of the following cases determine whether or not pure resonance occurs:

- (1) The function $F(t)$ has a period of 2π seconds and is defined as:

$$F(t) = \begin{cases} 8, & 0 < t < \pi, \\ -8, & -\pi < t < 0. \end{cases}$$

- (2) The function $F(t)$ has a period of 2 seconds and is defined as:

$$F(t) = \begin{cases} 8, & 0 < t < 1, \\ -8, & -1 < t < 0. \end{cases}$$

- (3) The function $F(t)$ has a period of 2π seconds and has the Fourier Series:

$$F(t) = \sum_{n \text{ even}} \frac{(-1)^n}{n^4} \sin nt.$$

Solution.

The natural frequency of the system is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{18/2} = 3$.

- (1) The function $F(t)$ is odd 2π -periodic. Therefore its Fourier Series is:

$$F(t) \sim \sum_{n=1}^{\infty} b_n \sin nt.$$

Since $nt = 3t = \omega_0 t$ for $n = 3$, we need to check if $b_3 = 0$. We have:

$$\begin{aligned} b_3 &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin(3t) dt = \frac{2}{\pi} \int_0^{\pi} F(t) \sin(3t) dt = \\ &= \frac{2}{\pi} \int_0^{\pi} 8 \sin(3t) dt = \frac{16}{3\pi} [-\cos(3t)]_0^{\pi} = \\ &= \frac{16}{3\pi} [-\cos(3\pi) + 1] = \frac{16}{3\pi} [-(-1) + 1] = \frac{32}{3\pi} \neq 0. \end{aligned}$$

Thus $b_3 \neq 0$ and hence pure resonance does occur.

- (2) The function $F(t)$ is odd 2-periodic. Therefore its Fourier Series is:

$$F(t) \sim \sum_{n=1}^{\infty} b_n \sin(\pi nt).$$

Recall that $\omega_0 = 3$. Since π is irrational, we have $\pi n \neq 3$ for $n = 1, 2, \dots$. Therefore pure resonance does not occur.

(3) The function $F(t)$ has a period of 2π seconds and has the Fourier Series:

$$F(t) = \sum_{n \text{ even}} \frac{(-1)^n}{n^4} \sin nt.$$

Since $n = 3$ is odd, the coefficient corresponding to $\sin 3t$ in the above decomposition of $F(t)$ is equal to 0.

Therefore pure resonance does not occur.

Some Trigonometric and hyperbolic trigonometric formulas

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}, \quad \cos^2 A = \frac{1 + \cos 2A}{2}.$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(-x) = \cosh(x), \quad \sinh(-x) = -\sinh(x),$$

$$\cosh(0) = 1, \quad \sinh(0) = 0,$$

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\cosh'(x) = \sinh(x), \quad \sinh'(x) = \cosh(x),$$

$$\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y),$$

$$\sinh(x - y) = \sinh(x) \cosh(y) - \cosh(x) \sinh(y).$$