

Problem 1.

Let $n \geq k > 0$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a smooth function such that for each $x \in M := f^{-1}(0)$ the derivative $f'(x)$ has rank k . Thus $f = (f_1, \dots, f_k)$, where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$.

Prove that for each $p \in M$ and any $v \in \mathbb{R}^n$ such that $v_p \in M_p$ we have

$$(\dagger) \quad \langle \text{grad}(f_i)|_p, v \rangle = 0, \quad i = 1, \dots, k.$$

Use this to deduce that for each $p \in M$ the tangent space M_p is exactly the set of all vectors $v_p \in \mathbb{R}^n_p$ such that v satisfies (\dagger) .

Here \langle, \rangle is the usual inner product on \mathbb{R}^n .

Problem 2.

Let M be the set of all $(x, y, z, w) \in \mathbb{R}^4$ satisfying the equations

$$\begin{aligned} x^2 - 2y + w + 1 &= 0 \\ e^x + w^2 + y^3 - z &= 0 \end{aligned}$$

Prove that M is a 2-manifold. For the point $p = (0, 1, 3, 1) \in M$ find a basis of M_p .

Problem 3.

Consider a singular 2-cube $c : [0, 1]^2 \rightarrow \mathbb{R}^3$ defined as

$$c(t, s) = (\sin(ts), t, t + s).$$

Let $\omega = 3y dx \wedge dz$ be the 2-form in \mathbb{R}^3 .

Compute $\int_c \omega$.

Problem 4.

Consider a singular 2-cube $c : [0, 1]^2 \rightarrow \mathbb{R}^3$ defined in Problem 3.

Compute ∂c .

Problem 5.

Let $M = \mathbb{S}^1$ be the unit circle in \mathbb{R}^2 .

Consider the following 1-form in \mathbb{R}^2

$$\omega_1(x, y) = \begin{cases} e^x dx - 2 dy, & \text{if } x^2 + y^2 \in \mathbb{Q} \\ -5x dx + xy dy & \text{if } x^2 + y^2 \notin \mathbb{Q}. \end{cases}$$

Define a 1-form ω on M by setting $\omega(x) := \omega_1(x)|_{M_x}$ for every $x \in M$.

Prove that ω is a smooth 1-form on M .

Problem 6.

Let $M \subseteq \mathbb{R}^n$ be a k -manifold. Suppose there exists a smooth k -form ω on M such that for every $x \in M$ we have $\omega(x) \neq 0$. Prove that M is orientable.