Math 302, Section B1
Challenge Problem no. 2

Assume that you are working in the setting neutral geometry.

Suppose $\triangle ABC$ is a triangle and that we chose a coordinate system on the line $\overleftrightarrow{BC}$ so that the coordinate of B is equal to 0 and the coordinate of C is equal to the distance $BC > 0$.

For any real number $t, 0 < t < BC$ let B_t be the point on the line $\overleftrightarrow{BC}$ with coordinate t and put $f(t) = m\angle BAB_t$ (see Figure 1 below).

Prove that the function $f(t)$ is continuous on the interval $0 < t < BC$.

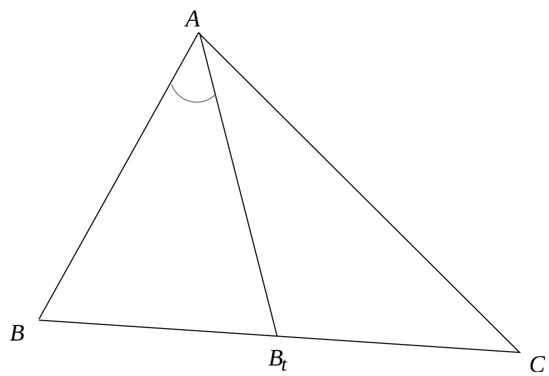

Figure 1:

Solution

Note that by the Angle Addition Postulate $f(t)$ is an increasing function of t, that is whenever $0 < t < t' < BC$ then $f(t) < f(t')$.

Suppose $f(t)$ is not continuous on the interval $(0, BC)$. Then there is some $t_0, 0 < t_0 < BC$ such that $f(t)$ is not continuous at t_0. By the properties of monotone functions [you need to look this up in a good calculus textbook] this implies that $f(t)$ "makes a jump" at t_0: that is, the limits on the left and the right

$$\alpha = \lim_{t \to t_0 -} f(t) \text{ and } \beta = \lim_{t \to t_0 +} f(t)$$

both exist but $\alpha < \beta$. The portion of the graph $y = f(t)$ near $t = t_0$ is shown in Figure 2.

The value $f(t_0)$ is distinct from at least one of α, β. Assume that $f(t_0) \neq \alpha$, as the case $f(t_0) \neq \beta$ is analogous. Thus $\alpha < f(t_0) \leq \beta$. Since $f(t)$ is increasing, the definition of α and β implies that for any $t < t_0$ we have $f(t) \leq \alpha$ and for any $t \geq t_0$ we have $f(t) \geq f(t_0)$. Thus for any $y \in (\alpha, f(t_0))$ the function $f(t)$ never achieves the value y. Pick any y' such that $\alpha < y' < f(t_0)$.

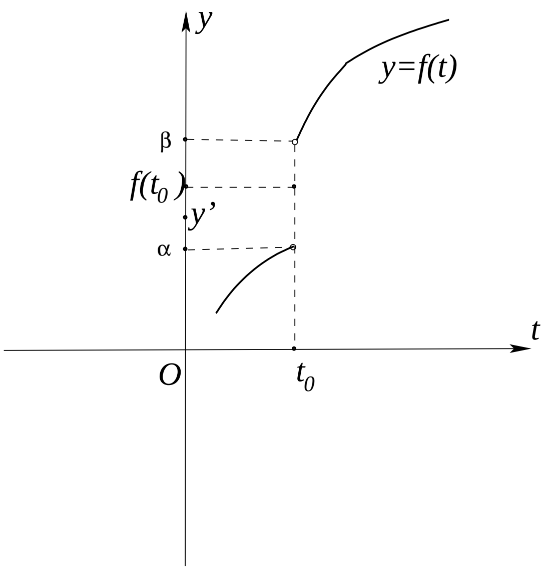

Figure 2: A portion of the graph $y = f(t)$ near $t = t_0$

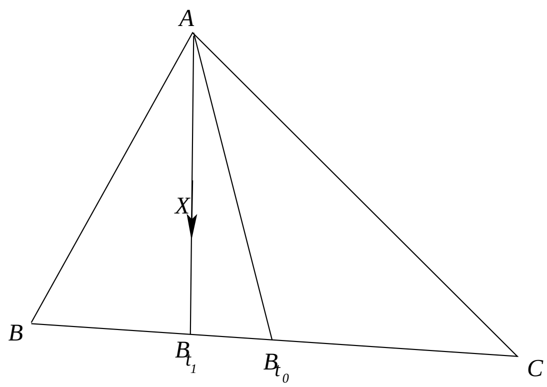

Figure 3:

By the Angle Construction Axiom there exists a ray $\overrightarrow{AX}$ (with X on the same side from $\overleftrightarrow{AB}$ as the point C) such that $m\angle BAX = y'$. Since $y' < m\angle BAB_{t_0} = f(t_0)$, the Cross-bar Theorem implies that the ray $\overrightarrow{AX}$ intersects the segment $\overline{BB_{t_0}}$ at some point B_{t_1} with $B - B_{t_1} - B_{t_0}$. Thus $0 < t_1 < t_0$ and $f(t_1) = m\angle BAB_{t_1} = y'$. This contradicts the choice of y' (since by that choice the function $f(t)$ never assumes the value y').

Thus the assumption that $f(t)$ was not continuous on $(0, BC)$ is false, and so $f(t)$ is in fact continuous on $(0, BC)$, as required.