Math 302, Section B1
Challenge Problem no. 4 (with solution)

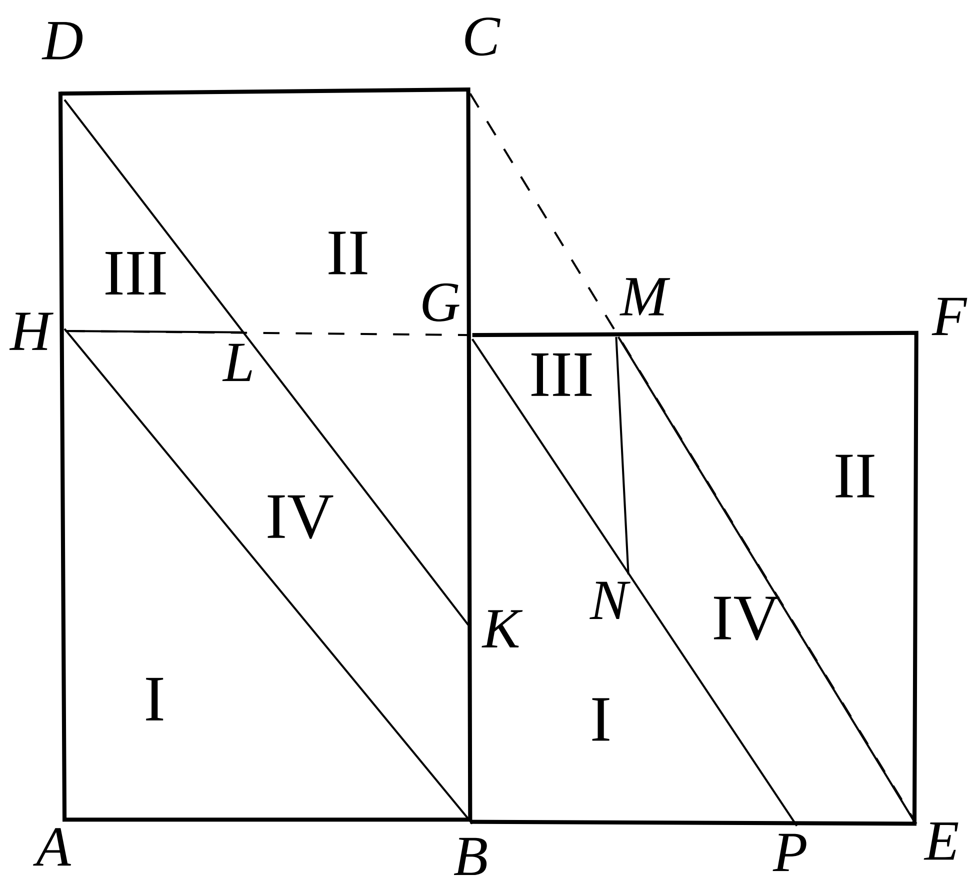
Prove that in Euclidean geometry any two rectangles of the same area are equivalent.

You may assume the standard facts regarding Euclidean geometry as known (e.g. formulas for the areas of rectangles, triangles, parallelograms etc).

Solution.

Since equivalence of polygons is transitive and symmetric, it suffices to show that every rectangle is equivalent to a square of the same area.

Let $ABCD$ be a rectangle. We may assume that $ABCD$ is not a square (otherwise there is nothing to prove) and that $b = AD > a = AB$. Construct a square $BEFG$ with the same area as $ABCD$ next to $ABCD$, as shown in the figure below. Denote $c = BE = BG$. Thus $c^2 = ab$ and so $c = \sqrt{ab}$.



Let M be the point of intersection of segments $\overline{CE}$ and $\overline{GF}$. We claim that $MF = a$. Indeed, $m\angle EMF = m\angle BEC$ by the converse of the Alternate Interior Angle Theorem (which does hold in the Euclidean geometry) and therefore the triangles $\triangle BCE$ and $\triangle MFE$ are similar. Hence $\frac{b}{c} = \frac{c}{MF}$ and so $MF = \frac{c^2}{b} =$

$\frac{ab}{b} = a$. Thus $MF = a$, as claimed. Draw lines parallel to $\overleftrightarrow{CE}$ through G, D, H, as shown in the figure. Thus $GMEP$ is a parallelogram and so $GP = ME$. Since $PE = GM = c - a$, we have $BP = c - PE = c - (c - a) = a$. Moreover $\angle GPB \cong \angle HBA$ by the converse of the Alternate Interior Angle Theorem. Hence by the AAS condition $\triangle HBA \cong \triangle GPB$. In particular $AH = BG = c$.

Similarly $DC = MF = a$, $\angle CDK \cong \angle EMF$ and hence by the AAS condition $\triangle DCK \cong \triangle MFE$.

Since $AH = c$, the points H, G, M, F lie on the same line. Let L be the intersection point of $\overline{HG}$ and $\overline{DK}$. Construct a ray making the right angle at M as shown and denote by N the intersection point of this ray with $\overline{GP}$. Thus $m\angle GMN = 90°$.

Then $DH = b - c = GC$, $\angle HLD \cong \angle GMC$ and so by the AAS condition $\triangle HDL \cong \triangle CGM$. The right triangles $\triangle CGM$ and $\triangle GMN$ share the side GM and $\angle MGN \cong \angle GMC$. Hence, again by AAS, $\triangle CGM \cong \triangle GMN$. Thus $CG = MN = b - c$ and $\triangle HDL \cong \triangle CGM \cong \triangle GMN$.

We claim that the quadrilaterals $HLKB$ and $MNPE$ are congruent. Indeed $HB = GP = ME$ and $HL = GM = PE$. Moreover $MN = CG = DH = KB$. Additionally $\angle EMN \cong \angle ECB \cong \angle AHB \cong \angle HBK$. Also, $\angle MEP \cong \angle HBA \cong \angle BHL$. Thus $HB = ME, HL = PE, KB = MN$ and $\angle LHB \cong \angle MEP, \angle KBH \cong \angle EMN$.

Hence the quadrilaterals $HBKL$ and $EMNP$ are congruent by the SASAS condition. We already know that $\triangle HBA \cong \triangle GPB$, $\triangle DCK \cong \triangle MFE$, $\triangle DHL \cong \triangle GMN$ and, also, $HBKL \cong EMNP$.

Therefore $ABCD$ is equivalent to $BEFG$, as required.