Math 302, Section B1; February 28, 2003
Exam 1 (with solutions)

Problem 1.

Consider the axiomatic system with undefined terms As, Bs and "in" and with the following axioms:

A1 There are at most two As in the system and there is at least one A in the system.

A2 For any B there are at least three distinct A's in it.

(1) Is this axiomatic system consistent? Explain why or why not.

Solution.

Yes, the system is consistent. We will establish consistency by producing a concrete model for the system.

In this model, which we will call Model 1, we interpret As as balls and B as urns. Model 1 has two balls and no urns.

Clearly, axioms A1 ans A2 are satisfied, so that Model 1 is a valid model for the axiomatic system and hence the system is consistent.

(2) Consider the following extra axiom:

A3 There are exactly two As in the system.

Is this axiom independent of the axioms A1-A2? Explain why or why not.

Solution.

Yes, axiom A3 is independent from axioms A1-A2. To establish independence of A3 with a consistent axiomatic system A1-A2 we must produce two models: one model which satisfies the axioms A1-A3 and another model which satisfies the axioms A1-A2 but does not satisfy A3.

Model 1 describes above satisfies all three axioms A1-A3.

Consider the following model, which we will call Model 2. Again, in Model 2 As stand for balls and Bs stand for urns. Model two has one ball and no urns.

Then Model 2 satisfies axioms A1-A2 but does not satisfy axiom A3.

Problem 2.

Assume that you are working in the SMSG setting of the Euclidean geometry. Consider the following relation on the set of all triangles. We say that a triangle $\triangle_1$ is *compatible* with a triangle $\triangle_2$ if some side of $\triangle_1$ is congruent to some side of $\triangle_2$.

Is the relation of being compatible an equivalence relation on the set of all triangles? Explain why or why not.

Solution.

The relation of being compatible is not transitive and therefore it is not an equivalence relation.

Indeed, in the Euclidean plane we can find three triangles $\triangle_1$, $\triangle_2$ and $\triangle_3$ such that:

(a) triangle $\triangle_1$ has sides of lengths $1, 1, 1$;
(b) triangle $\triangle_2$ has sides of lengths $1, 2, 2$
(c) triangle $\triangle_2$ has sides of lengths $2, 2, 2$.

Then $\triangle_1$ is compatible with $\triangle_2$ and, also, $\triangle_2$ is compatible with $\triangle_3$. However, $\triangle_1$ is not compatible with $\triangle_3$.

Problem 3.

Explain how SMSG Postulate 4 follows from SMSG Postulates 2 and 3.

Solution.

Let l be a line and let P, Q be two distinct points on this line. We mush derive from SMSG Postulates 2 and 3 that there exists a coordinate system on l such that the coordinate of P is equal to zero and the coordinate of Q is positive.

By Postulate 2 there exists some coordinate system on l, that is a bijective function $x : l \to (-\infty, \infty)$ such that for any points A, B on l we have $d(A, B) = |x(A) - x(B)|$.

Define a new function $y : l \to (-\infty, \infty)$ by formula:

$$y(A) := x(A) - x(P) \text{ for any point } A \in l.$$

Then y is a new coordinate system on l. The new coordinate of P is $y(P) = x(P) - x(P) = 0$. If $y(Q) > 0$, then we are done: the coordinate system y has all the required properties.

Suppose now that $y(Q) < 0$. We then define a new function $z : l \to\to (-\infty, \infty)$ by formula:

$$z(A) := -y(A) \text{ for any point } A \in l.$$

Then z is a new coordinate system on l. Moreover, the new coordinates of P and Q are:

$$z(P) = -y(P) = -0 = 0$$
$$z(Q) = -y(Q) > 0.$$

So the coordinate system z satisfies all the required properties.

Problem 4.

In the setting of neutral geometry state and prove the ASASA congruence theorem for quadrilaterals.

Solution.

Theorem. If the vertices of two quadrilaterals are in one-to-one correspondence such that the three consecutive angles and the two included sides of one quadrilateral are congruent to the corresponding angles and sides the other quadrilateral, then the quadrilaterals are congruent.

Proof.

Suppose quadrilaterals $ABCD$ and $A'B'C'D'$ are such that $\angle DAB \cong \angle D'A'B'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$, $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$. We must show that the quadrilateral $ABCD$ is congruent to the quadrilateral $A'B'C'D'$. That it, by definition of congruence for quadrilaterals, it remains for us to establish that $\overline{AD} \cong \overline{A'D'}$, $\overline{CD} \cong \overline{C'D'}$ and $\angle CDA \cong C'D'A'$.

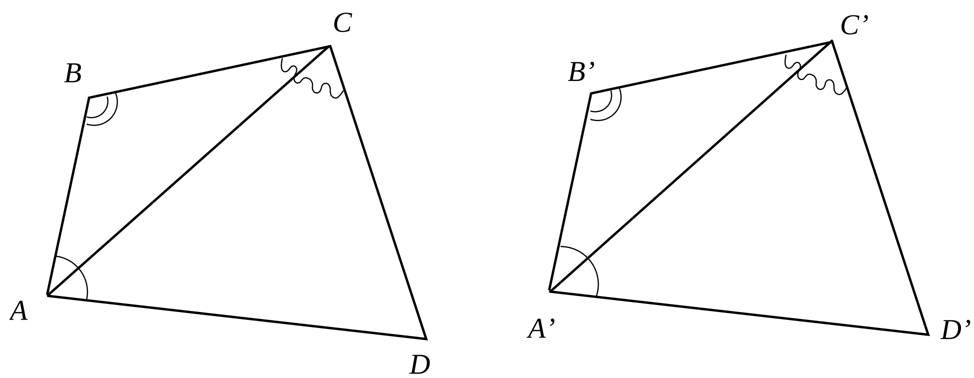

Figure 1: Figure for Problem 4.

Draw the segments $\overline{AC}$ and $\overline{A'C'}$. Then by the SAS axiom $\triangle ABC \cong \triangle A'B'C'$. Therefore $\overline{AC} \cong \overline{A'C'}$, $\angle BAC \cong \angle B'A'C'$ and $\angle BCA \cong \angle B'C'A'$. By the angle addition axiom $m\angle CAD = m\angle BAD - m\angle BAC$ and $m\angle C'A'D' = m\angle B'A'D' - m\angle B'A'C'$. By assumption $m\angle BAD = m\angle B'A'D'$. We have just concluded that $m\angle BAC = m\angle B'A'C'$. Therefore $m\angle CAD = m\angle C'A'D'$ and so $\angle CAD \cong \angle C'A'D'$.

Similarly, the angle addition axiom implies that

$$m\angle ACD = m\angle BCD - m\angle BCA = m\angle B'C'D' - m\angle B'C'A' = m\angle A'C'D',$$

and so $\angle ACD \cong \angle A'C'D'$. Therefore by the ASA Congruence Theorem for triangles $\triangle ACD \cong \triangle A'C'D'$. This implies that $\overline{AD} \cong \overline{A'D'}$, $\overline{CD} \cong \overline{C'D'}$ and $\angle CDA \cong C'D'A'$.

Thus the four sides and the four angles of $ABCD$ are congruent to the corresponding sides and angles of $A'B'C'D'$ and hence the quadrilaterals $ABCD$ and $A'B'C'D'$ are congruent as required.

Problem 5.

In the setting of neutral geometry prove that there does not exist a triangle two of whose angles have measures $75°$ and $140°$.

Solution.

Suppose there exists a triangle $\triangle ABC$ such that $m\angle BAC = 75°$ and $m\angle BCA = 140°$.

Consider a point D on the ray $\overrightarrow{AC}$ such that $A - C - D$. Then the angles $\angle BCD$ and $\angle BCA$ form a linear pair and hence are supplementary by SMSG Postulate 14. Therefore $m\angle BCD = 180° - 140° = 40°$.

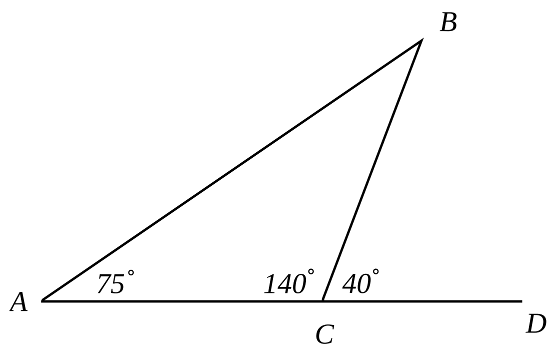

Figure 2: Figure for Problem 5.

The angle $\angle BCD$ is an exterior angle for the triangle $\triangle ABC$ and therefore by the Exterior Angle Theorem we must have $m\angle BCD > m\angle BAC$, that is $40° > 75°$, which yields a contradiction.