Math 302, Section B1; April 17, 2003
Exam 2 (with solutions)

Problem 1.

Prove that in hyperbolic geometry if two Saccheri quadrilaterals have congruent bases and congruent summit angles then these quadrilaterals are congruent.

Solution.

Suppose $ABCD$ and $EFGH$ are Saccheri quadrilaterals with congruent bases and congruent summit angles (see Figure). Thus $AB = EF$ and $\angle BAD \cong \angle DCA \cong \angle FGH \cong \angle EHG$.

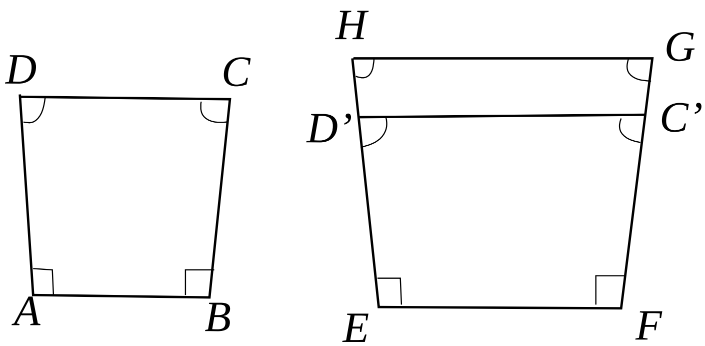


Figure 1: Illustration for Problem 1.

We need to show that $ABCD \cong EFGH$.

If $AD = EH$ then $CB = GF$ and the quadrilaterals $ABCD$ and $EFGH$ are congruent by definition of congruence for quadrilaterals.

Suppose now $AD \neq EH$. Without loss of generality we may assume $AD < EH$, so that $BC < FG$. Choose points D'', C' so that $E - D' - H$, $F - C' - G$ and $ED' = AD = BC = FC'$. Draw the segment $\overline{D'C'}$. Then the quadrilaterals $DABC$ and $D'EFC'$ are congruent by the $SASAS$ congruence condition since by assumption $AB = EF$. Hence $\angle ADC \cong \angle ED'C'$ and $\angle BCD \cong \angle FC'D'$. Denote $\alpha = m\angle ADC$.

Then the quadrilateral $HD'C'G$ has angle sum

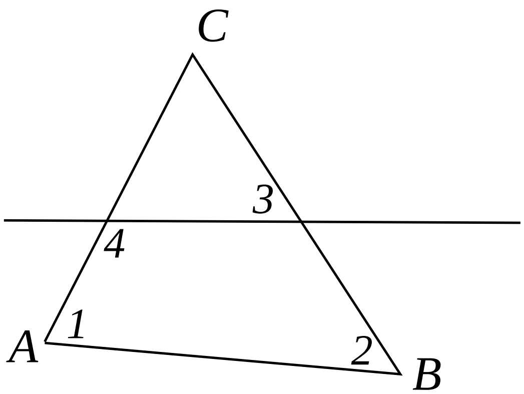
$$\alpha + \alpha + (180° - \alpha) + (180° - \alpha) = 360°.$$

However, by Corollary 6.4.2 in hyperbolic geometry every convex quadrilateral has angle sum smaller than 360°, yielding a contradiction.

Problem 2.

In the setting of hyperbolic geometry consider the triangle $\triangle ABC$ shown in the figure below. Suppose that $m\angle 1 = 15°$, $m\angle 2 = 30°$, $m\angle 3 = 70°$, $m\angle 4 = 100°$ and $d(\triangle ABC) = 116°$. Find $m\angle ACB$.

Solution.



We have

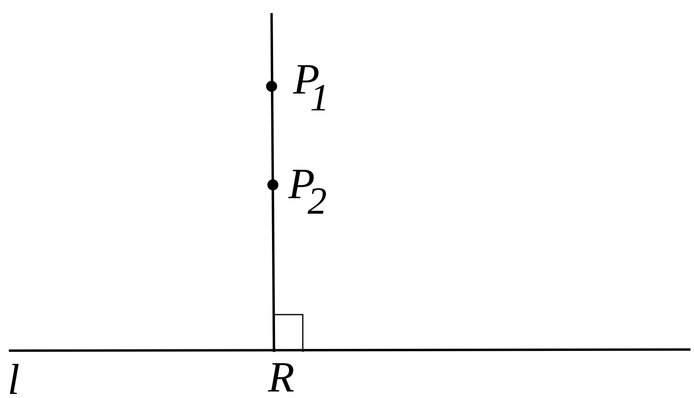
$$116^\circ = d(\triangle ABC) = 180^\circ - m\angle 1 - m\angle 2 - m\angle ACB =$$
$$= 180^\circ - 15^\circ - 30^\circ - m\angle ACB = 135^\circ - m\angle ACB$$

and therefore

$$m\angle ACB = 135^\circ - 116^\circ = 19^\circ.$$

Problem 3.

In the settings of neutral geometry let l be a line and let P_1, P_2 be points on a ray perpendicular to l lying on the same side of l in the plane and such that $d(P_1, l) > d(P_2, l) > 0$ (see Figure below). Prove that the measure of the angle of parallelism for P_1 and l is no greater than the measure of the angle of parallelism for P_2 and l.



Solution.

Suppose, on the contrary that $AP(P_1, l) > AP(P_2, l)$. Denote $d_1 = AP(P_1, l)$ and $d_2 = AP(P_2, l)$. Let Q_1 and Q_2 be points on the same side of $\overleftrightarrow{P_1P_2}$ such that $m\angle RP_1Q_1 = d_1$ and $m\angle RP_2Q_2 = d_2$.

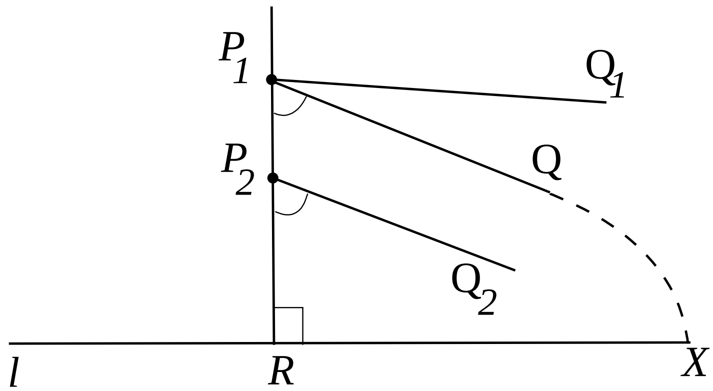


Figure 2: Illustration for the solution of Problem 3

Since by assumption $d_1 > d_2$, we can choose a point Q on the same side of $\overleftrightarrow{P_1P_2}$ as Q_1, Q_2 such that $m\angle RP_1Q = d_2 = m\angle RP_2Q_2$. Then $\overleftrightarrow{P_1Q}||\overleftrightarrow{P_2Q_2}$ by the Alternate Interior Angle Theorem. By part (i) of Theorem 6.2.1 we have $\overleftrightarrow{P_1Q}||l$.

Since $m\angle RP_1Q < AP(P_1, l)$, part (ii) of Theorem 6.2.1 implies that $\overrightarrow{P_1Q} \cap l \neq \emptyset$. Denote $X = \overrightarrow{P_2Q_2} \cap l$. Consider the triangle $\triangle RP_1X$. By Theorem 3.2.5 (Pasch's Axiom) the line $\overrightarrow{P_2Q_2}$ must intersect at least one of the segments $\overline{P_1Q}$, $\overline{RX}$. But we have shown that $\overrightarrow{P_2Q_2}$ is parallel to both $\overrightarrow{P_1Q} = \overrightarrow{P_1X}$ and $\overrightarrow{RX} = l$, yielding a contradiction.

Problem 4.

For each of the following statements indicate if it is true or false. You do not need to give reasons for your answers in this problem.

1. If there exists a triangle with angle sum less than $180°$ then for any line l and any point P not on l there is at most one line through P parallel to l. [FALSE]

2. In hyperbolic geometry the measure of an angle in an equilateral triangle is less than $60°$. [TRUE]

3. In hyperbolic geometry every triangle can be circumscribed. [FALSE]

4. In hyperbolic geometry two triangles are equivalent if and only if they have the same defect. [TRUE]

5. In hyperbolic geometry Lambert quadrilaterals do not exist. [FALSE]

Problem 5.

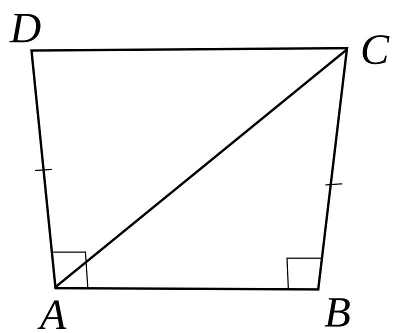
Prove the following statement (this is Theorem 6.3.7 in the textbook):

In hyperbolic geometry the summit of a Saccheri quadrilateral is longer then the base.

Solution 1. See the proof of Theorem 6.3.7 on page 307 in the book.

Solution 2.

Let $ABCD$ be a Saccheri quadrilateral in hyperbolic geometry with base $\overline{AB}$ and legs $\overline{BC}$ and $\overline{AD}$. By Theorem 3.6.6 the summit is greater than or equal to in length than the base, that is $CD \geq AB$. If $CD > AB$, there is nothing to prove. Suppose $CD = AB$. Draw a diagonal $\overline{AC}$.



Then $\triangle ABC \cong \triangle ACD$ by the SSS congruence condition for triangles. Hence $m\angle ADC = m\angle ABC = 90°$. This contradicts Theorem 3.6.2 asserting that in hyperbolic geometry the summit angles of any Saccheri quadrilateral are acute.