Solutions to H/wk 4

Ch 3.2 no. 1.

(a) A point A is said to lie *between* points A and B, denoted $A - C - B$, if A, B, C are distinct and $d(A,B) = d(A,C) + d(C,B)$.

(b) For two distinct points A, B a *line segment* $\overline{AB}$ consists of the points A, B and of all points C on the unique line passing through A and B such that $A - C - B$. The points A and B are called the *endpoints* of $\overline{AB}$.

(c) If A, B, C are three distinct non-colinear points, then the *angle* $\angle BAC$ consists of the points on rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ and of all points D that lie between a point on $\overrightarrow{AB}$ and a point on $\overrightarrow{AC}$. The rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ are called the *sides* of $\angle BAC$ and the point A is called the *vertex* of $\angle BAC$.

(d) An angle is *obtuse* if it has measure more than $90°$. An angle is *acute* if it has measure less than $90°$. An angle is *right* if it has measure $90°$.

(e) Two angles are *adjacent* of they have a common side and their intersection is equal to that side.

(f) Two angles are *vertical* if they have a common vertex, their intersection is equal to that vertex and if the sides of each angle can be ordered in such a way that the union of the first sides of these angles is a line and the union of the second sides of these angles is a line.

(g) Two angles are *supplementary* if the sum of their measures is $180°$. Two angles are *complementary* if the sum of their measures is $90°$.

(h) The midpoint of a line segment is a point on the segment that is equidistant from the endpoints of the segment.

(i) A *bisector* of an angle $\angle BAC$ is a ray $\overrightarrow{AD}$ that is contained in $\angle BAC$ and such that $m\angle BAD = m\angle DAC = \frac{1}{2} m\angle BAC$.

(j) Two lines l_1, l_2 are said to be *perpendicular* if they intersect at a point A such that for any point B on l_1 and any point C on l_2 such that $B \neq A$ and $C \neq A$ the angle $\angle BAC$ is right.

(k) If A, B, C are three non-colinear points then a *triangle* $\triangle ABC$ is the intersection of the angles $\angle ABC$, $\angle BAC$ and $\angle CBA$. The points A, B, C are called *vertices* of $\triangle ABC$ and the segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ are called the *sides* of $\triangle ABC$.

(l) Let $A_1, A_2, A_3, \dots A_n$ (where $n \geq 4$) be n distinct points in the plane.

Suppose that:

(1) No three distinct points in the list $A_1, A_2, A_3, \dots A_n$ are colinear; and
(2) For each $i = 2, \dots, n-2$ the intersection of the triangles $\triangle A_1 A_i A_{i+1}$ and $A_1 A_{i+1} A_{i+2}$ is equal to the segment $\overline{A_1 A_{i+1}}$; and
(3) The union of the triangles $\triangle A_1 A_2 A_3, \triangle A_1 A_3 A_4, \dots \triangle, A_1 A_{n-1} A_n$ is convex.

Then a *polygon* $P(A_1, \dots, A_n)$ is the union of triangles

$$\triangle A_1 A_2 A_3, \triangle A_1 A_3 A_4, \dots, \triangle A_1 A_{n-1} A_n.$$

The points $A_1, \dots, A_n$ are called the *vertices* of $P(A_1, \dots, A_n)$ and the segments $\overline{A_1 A_2}, \overline{A_2 A_3}, \dots, \overline{A_{n-1} A_n}, \overline{A_n A_1}$ are called the *sides* of $P(A_1, \dots, A_n)$.

(m) The *interior* of an angle (triangle, polygon) consists of all those points of the angle (triangle, polygon) that do not lie on its sides.

Ch 3.2 no. 4.

We will show that the relation of angle congruence is an equivalence relation. Recall that the angles $\langle ABC$ and $\angle A'B'C'$ are said to be *congruent*, denoted $\langle ABC \cong \angle A'B'C'$, if $m\angle ABC = m\angle A'B'C'$.

(1) **Reflexivity.** For any angle $\angle ABC$ we have $m\angle ABC = m\angle ABC$ and hence $\angle ABC \cong \angle ABC$.

(2) **Symmetry.** Suppose $\angle ABC \cong \angle A'B'C'$.

Then $m\angle ABC = m\angle A'B'C'$. Therefore $m\angle A'B'C' = m\angle ABC$ and hence $\langle A'B'C' \cong \angle ABC$.

(3)**Transitivity**. Suppose $\langle ABC \cong \angle A'B'C'$ and $\langle A'B'C' \cong \angle A''B''C''$.

Then $m\angle ABC = m\angle A'B'C'$ and $m\angle A'B'C' = m\angle A''B''C''$. Hence $m\angle ABC = m\angle A''B''C''$ and so $\langle ABC \cong \angle A''B''C''$

Ch 3.2 no. 6.

We need to prove that supplements and complements of congruent angles are congruent. We will do that for complements.

Suppose α, β are angles such that $\alpha \cong \alpha'$ and let β be a complementary angle of α and β' be a complementary angle of α'. We need to show that $\beta \cong \beta'$.

Since $\alpha \cong \alpha'$, we have $m\alpha = m\alpha'$. Since β is a complement of α and β' is a complement of α', we have $m\beta = 90^\circ - m\alpha$ and $m\beta' = 90^\circ - m\alpha'$. Hence

$$m\beta = 90^\circ - m\alpha = 90^\circ - m\alpha' = m\beta'.$$

Thus $m\beta = m\beta'$ and $\beta \cong \beta'$, as required.

Ch 3.2 no. 7. (Draw a picture for the argument below)

We need to prove that vertical angles are congruent.

Suppose angles $\angle BAC$ and $\angle B'AC'$ are vertical, so that $\overrightarrow{AB} \cup \overrightarrow{AB'}$ is a line l_1 and $\overrightarrow{AC} \cup \overrightarrow{AC'}$ is a line l_2.

Since the angles $\angle BAC$ and CAB' form a linear pair, SMSG Postulate 14 implies that these angles are supplementary and so $m\angle BAC + m\angle CAB' = 180^\circ$.

Similarly, the angles $\angle CAB'$ and $B'AC'$ form a linear pair and therefore by SMSG Postulate 14 these angles are supplementary and $m\angle CAB' + m\angle B'AC' = 180^\circ$.

Therefore

$$m\angle C'AB' = 180^\circ - m\angle CAB' = 180^\circ - (180^\circ - m\angle BAC) = m\angle BAC$$

Thus $m\angle C'AB' = m\angle BAC$ and so $\angle C'AB' \cong \angle BAC$, as required.

Ch 3.2 no. 9. (Draw a picture for the argument below)

We need to prove that if a point is on a perpendicular bisector of a line segment, then it is equidistant from the endpoints of that segment.

Let l be a perpendicular bisector to a segment $\overline{AB}$ and denote the midpoint of $\overline{AB}$ by T.

Suppose C is a point on l.

If $C = T$ then $d(C, A) = d(C, B) = \frac{1}{2}d(A, B)$ and C is equidistant from A and B. Suppose now that $C \neq T$, so that A, B, C are not co-linear.

We have $\angle CTA \cong \angle CTB$, since both angles are right. Moreover, $\overline{CT} \cong \overline{CT}$ and $\overline{TA} \cong \overline{TB}$ since $d(A, T) = d(B, T)$.

Therefore by the SAS axiom (SMSG Postulate 15) $\triangle CTA \cong \triangle CTB$. This implies that $\overline{AC} \cong \overline{BC}$ and so $d(A, C) = d(B, C)$, as required.