Math 317 Section B1 Exam 1 (with solutions)

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Problem 1.[20 points]

(1) Let a, b be even integers. Suppose there exist integers s, t such that $sa + tb = 2$. Prove that $gcd(a, b) = 2$.

Solution. Let $d = gcd(a, b)$. Then $d|a$ and hence $d|sa$. Similarly $2|b$ and hence $d|tb$. Therefore $d|(sa + tb)$ that is $t|2$. This implies that $d \leq 2$. Since a, b are even 2 is a common divisor of a and b. Thus $2 \leq gcd(a, b) = d$. So $2 \leq d \leq 2$ implies $d = 2$.

(2) Show that $\sqrt[3]{3}$ is irrational.

Solution. Suppose, on the contrary, that $\sqrt[3]{3}$ is rational. Then we can express $\sqrt[3]{3}$ as a fraction of two positive integers in lowers terms:

$$\sqrt[3]{3} = \frac{a}{b}, \text{ where } gcd(a, b) = 1, a > 0, b > 0.$$

Raising to the third power and cross-multiplying the above equation we get

$$3 = \frac{a^3}{b^3}, \quad 3b^3 = a^3.$$

Thus $3|a^3$. Since 3 is prime, Euclid's lemma implies that $3|a$. Thus $a = 3m$ for some integer $m > 0$. Therefore

$$3b^3 = (3m)^3 = 27m^3 \text{ and so } b^3 = 9m^3 = 3 \cdot 3m^3.$$

Thus $3|b^3$. By Euclid's lemma this again means that $3|b$. Thus $3|a$ and $3|b$ which contradicts the assumption that $(a, b) = 1$.

Problem 2.

Prove that for every integer $n \geq 2$

$$n = \sum_{r=1}^{n} \binom{n}{r} r 3^{n-r} 2^{r-1} (-1)^{r-1}$$

Solution.

By the Binomial Theorem we have:

$$(3 + x)^n = \sum_{r=0}^{n} \binom{n}{r} 3^{n-r} x^r$$

Differentiating both sides along x we obtain:

$$n(3 + x)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r 3^{n-r} x^{r-1}$$

(Note that the term corresponding to $r = 0$ was constant and hence vanishes after differentiation).

Substituting $x = -2$ in the above formula we get:

$$n = n(3 - 2)^n = \sum_{r=1}^{n} \binom{n}{r} r 3^{n-r} (-2)^{r-1} = \sum_{r=1}^{n} \binom{n}{r} r 3^{n-r} 2^{r-1} (-1)^{r-1}.$$

Problem 3.
(a) Find the general solution for the following system of congruence equations:

$$(1) \qquad \begin{cases} x \equiv 4 \mod 8 \\ 3x \equiv 1 \mod 5 \end{cases}$$

Solution.
First note that

$$3x \equiv 1 \mod 5 \iff 6x \equiv 2 \mod 5 \iff x \equiv 2 \mod 5$$

where the first equivalence is valid since $(2, 5) = 1$. Thus the original system is equivalent to:

$$(2) \qquad \begin{cases} x \equiv 4 \mod 8 \\ x \equiv 2 \mod 5. \end{cases}$$

From the first equation we have $x = 8k + 4$ for an integer k. Substituting this into the second equation, we get:

$$8k + 4 \equiv 2 \mod 5 \iff 8k \equiv -2 \mod 5 \iff 3k \equiv -2 \mod 5 \iff$$
$$3k \equiv 3 \mod 5 \iff 6k \equiv 6 \mod 5 \iff k \equiv 1 \mod 5.$$

Indeed, $k = 1$ and $x = 8 \cdot 1 + 4 = 12$ is a solution of the original congruence system (1) as well as of (2). Since $(8, 5) = 1$, by the Chinese Remainder Theorem applied to (2) the general solution of the system is

$$x \equiv 12 \mod 40.$$

(b) Find the general solution for the following congruence equation:

$$4x \equiv 1 \mod 10$$

Solution.
If x is an integer such that $4x \equiv 1 \mod 10$ then $4x - 1 = 10k$ for some integer k. Since $2|10$, the number $10k$ is even. However, the number $4x$ is even and hence $4x-1$ is odd. Thus $4x-1 = 10k$ is both even and odd, which is a contradiction. Hence the congruence $4x \equiv 1 \mod 10$ has no solutions.
Give all the details of your work.

Problem 4.
Let $\sigma \in S_n$ (where $n \geq 9$) be a permutation such that

$$\sigma = (123 \ldots 89)\beta\gamma$$

where $\alpha = (123 \ldots 89)$, β and γ are disjoint permutations.
(a) Find $\sigma^9(1)$.
(b) Suppose that β is a transposition and γ is a 4-cycle. Find $sgn(\sigma)$.
(c) Let $\delta = (12)(34) \in S_n$. Can δ and σ be conjugate in S_n? Explain why.

Solution.
(a) Since α, β, γ are disjoint and β, γ fix 1, we have $\sigma^k(1) = \alpha^k(1)$ for every k. Since α is a 9-cycle, $\alpha^9 = 1$ is the identity permutation and in particular $\alpha^9(1) = 1$. Hence $\sigma^9(1) = 1$.
(b) Recall that if ω is an r-cycle then $sgn(\omega) = (-1)^{r+1}$. By the multiplicative property of sgn we have

$$sgn(\sigma) = sgn(\alpha)sgn(\beta)sgn(\gamma) = (-1)^{10}(-1)^3(-1)^5 = 1 \cdot (-1) \cdot (-1) = 1.$$

(c) The permutations δ and σ have different cycle structure since the complete factorization of σ contains a 9-cycle while the the complete factorization of δ does not contain a 9-cycle. Hence δ and σ cannot be conjugate in S_n.
Problem 5.[20 points]
Use induction to prove that for every integer $n \geq 2$

$$n \geq \sqrt[3]{n+4}.$$

Solution.
Base of Induction.
For $n = 2$ $8 \geq 6$ and hence $2 \geq \sqrt[3]{6} = \sqrt[3]{2+4}$ as required.
Inductive Step.
Suppose $n \geq$ and $n \geq \sqrt[3]{n+4}$ is known to hold. We need to deduce that $n+1 \geq \sqrt[3]{(n+1)+4} = \sqrt[3]{n+5}$ also holds
From $n \geq \sqrt[3]{n+4}$ by raising both sides to the third power we get $n^3 \geq n+4$.
We have

$$n+1 \geq \sqrt[3]{n+5} \iff (n+1)^3 \geq n+5 \iff$$
$$n^3 + 3n^2 + 3n + 1 \geq n+5 \iff n^3 + 3n^2 + 3n \geq n+4$$

The last inequality follows from the inductive hypothesis $n^3 \geq n+4$ since $3n^2 + 3n \geq 0$.