Math 317 Section B1 Exam 2 (with solutions)
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Problem 1.[20 points]
(1) Show that for every group H the group $G = S_3 \times H$ has a subgroup of index 6.
(2) If H is a group, $G = S_3 \times H$ and $h \in H$, is it possible for the element $g = (\,(123), h)$ to have order 100 in G? If yes, give an example of such H and h. If not, explain why not.
Solution.
(1) Put

$$K := \{(1, h) \,|\, h \in H\} \subseteq S_3 \times H.$$

Then it is easy to see that K is a subgroup of $G = S_3 \times H$ of order $|H|$ (and in fact K is isomorphic to H). Hence by Largange's Theorem

$$[G : K] = \frac{|G|}{|K|} = \frac{|S_3| \cdot |H|}{|H|} = 6.$$

(2) Note that the 3-cycle $(123) \in S_3$ has order 3. Hence $(123)^{100} = (123)^{99+1} = (123)^1 = (123)$. Therefore for $g = (\,(123), h)$ we have $g^{100} = ((123)^{100}, h^{100}) = ((123), h^{100}) \neq (1, 1)$. Thus g cannot have order 100.
Problem 2.[20 points]
Suppose $H \leq G$ and suppose there is an element $g \in G$ such that for every $n \neq 0$ we have $g^n \notin H$. Prove that H has infinite index in G.
Solution.
Suppose on the contrary, that H has finite index in G. Then the family of cosets $\{g^n H \,|\, n > 0\}$ has only finitely many distinct cosets and hence has some repetitions. Therefore there exist $0 < i < j$ such that $g^j H = g^i H$. Hence there is $h \in H$ such that $g^j = g^i h$. This implies that $g^{j-i} = h \in H$, which contradicts our assumptions about g since $j - i \neq 0$.
Problem 3.[20 points]
(1) Are the groups $S_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_8$ isomorphic? If not, explain why. If yes, produce an isomorphism between them.
(2) Are the groups A_5 and S_4 isomorphic? If not, explain why. If yes, produce an isomorphism between them.
Solution.
(1) The groups $S_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_8$ are not isomorphic since the group $\mathbb{Z}_3 \times \mathbb{Z}_8$ is abelian while the group $S_3 \times \mathbb{Z}_4$ is not abelian [For example the elements $((12), [0])$ and $((13), [0])$ do not commute in $S_3 \times \mathbb{Z}_4$].
(2) The groups A_5 and S_4 are not isomorphic since they have different orders:

$$|A_5| = \frac{1}{2}5! = 60, \quad |S_4| = 4! = 24.$$

Problem 4.[20 points]

Let f be a homomorphism $f : G \to H$, such that $|H| = 150$. Suppose $K \leq G$ is a subgroup such that $|K| = 45$ and that $K \cap ker(f)$ is a cyclic group of order 3. Find the index of the subgroup $f(K)$ in H.

Solution.

By the 1-st Isomorphism Theorem $f(K) \cong K/(K \cap ker(f))$. Hence by Lagrange's Theorem

$$|f(K)| = \frac{|K|}{|K \cap ker(f)|} = \frac{45}{3} = 15.$$

Again by Lagrange's Theorem we have:

$$[H : f(K)] = \frac{|H|}{|f(K)|} = \frac{150}{15} = 10.$$

Problem 5.[20 points]

Let $\mathcal{F}$ be the ring consisting of all functions $f : \mathbb{R} \to \mathbb{R}$.

Let $S = \{f \in \mathcal{F} \mid f(0) = f(1)\}$. Let $T = \{f \in \mathcal{F} \mid f(0) + f(1) = 0\}$.

Is S a subring of $\mathcal{F}$? Is T a subring of $\mathcal{F}$?

Solution.

The set S is a subring of $\mathcal{F}$. Indeed, if $f, g \in S$, that is $f(0) = f(1), g(0) = g(1)$ then

$$(fg)(0) = f(0)g(0) = f(1)g(1) = (fg)(1), \text{ and}$$
$$(f-g)(0) = f(0) - g(0) = f(1) - g(1) = (f-g)(1),$$

and therefore $fg \in S$, $f - g \in S$. Also, the constant function 1 clearly belongs to S.

The set T is not a subring of $\mathcal{F}$ since the constant function 1 does not belong to T. Also, T is not closed under multiplication. For example Let $f(x) = 1 - 2x$. Then $f(0) = 1$, $f(1) = -1$ and hence $f \in T$. However $f^2 = f \cdot f \notin T$. Indeed, $f^2(0) = f^2(1) = 1$ and $f^2(0) + f^2(1) = 1 + 1 = 2 \neq 0$.