Math 385 Midterm Exam 1 (SOLUTIONS)

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Problem 1. [20 points] In this problem you need to mark each option as either TRUE or FALSE. No explanations of your answers are necessary.

(a) [5 points] The differential equation $y' = 2xy + 1$

(1) [1 point] is separable. [FALSE]

(2) [1 point] is homogeneous (in the sense of Ch. 1.6). [FALSE]

(3) [1 point] is linear [TRUE]

(4) [2 points] has at most one solution on the interval $(-\infty, \infty)$ [FALSE]

(b) [6 points]

(1) [2 points] The substitution $v = y/x$ transforms a homogeneous equation into a separable equation. [TRUE]

(2) [2 points] The substitution $v = y^{1-n}$ (for an appropriate n) transforms a Bernoulli equation into a separable equation. [FALSE]

(3) [2 points] The substitution $v = ax + by + c$ transforms an equation $y' = F(ax + by + c)$ (where $b \neq 0$) into a linear first-order equation. [FALSE]

(c)[9 points]

(1) [3 points] The problem

$$y'' + 2y' - 4y = 0, y(0) = 1$$

has infinitely many solutions on $(-\infty, \infty)$. [TRUE]

(2) [3 points] The functions

$$f(x) = |x|, g(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

are linearly dependent on $(-\infty, \infty)$. [TRUE]

(3) [3 points] Whenever $f(x)$ and $g(x)$ are functions such that $W(f, g)(0) = 0$ then $f(x)$ and $g(x)$ are linearly dependent on $(-\infty, \infty)$. [FALSE]

Problem 2. [20 points] Solve the following initial value problem on $\mathbb{R}$:

$$y'' + 6y' + 9y = 0, y(0) = 0, y'(0) = 2.$$

Solution.

We first find the general solution of the equation $y'' + 6y' + 9y = 0$ on $(-\infty, \infty)$. The characteristic equation is:

$$\lambda^2 + 6\lambda + 9 = 0, \quad (\lambda + 3)^2 = 0,$$

and hence the general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}, \qquad \text{where } c_1, c_2 \in \mathbb{R}.$$

Differentiating this formula we get

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}.$$

Using the conditions $y(0) = 0, y'(0) = 2$ we get:

$$c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 0, \quad -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0 = 2$$
$$c_1 = 0, \quad -3c_1 + c_2 = 2$$
$$c_1 = 0, \quad c_2 = 2.$$

Hence the solution of the original initial value problem is $y = 2xe^{-3x}$.

Problem 3. [20 points] Solve the following equation

$$y' = \frac{x + y}{x - y}$$

assuming $x > 0$.

Solution.

Dividing both the numerator and the denominator in the right-hand side by x we get

$$\frac{dy}{dx} = \frac{1 + y/x}{1 - y/x}.$$

This is a homogeneous equation in the sense of Ch. 1.6. We use the substitution $v = y/x$. Hence

$$y = vx, \frac{dy}{dx} = \frac{dv}{dx} x + v.$$

Thus after the substitution $v = y/x$ the equation becomes

$$\frac{dv}{dx} x + v = \frac{1 + v}{1 - v}$$
$$\frac{dv}{dx} x = \frac{1 + v}{1 - v} - v = \frac{1 + v^2}{1 - v}$$
$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x}.$$

This is a separable equation. Hence we get

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$$
$$\int \frac{1}{1+v^2}\,dv - \frac{1}{2}\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x}$$
$$\arctan v - \frac{1}{2}\ln(1+v^2) = \ln x + C.$$

Using $v = y/x$ we get

$$\arctan \frac{y}{x} - \frac{1}{2}\ln(1+\frac{y^2}{x^2}) = \ln x + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

This is the general solution, given in implicit form, of the original differential equation.

Problem 4. [20 points] Verify that the following equation is exact and then solve it:

$$(*) \qquad (x^3 + \frac{y}{x})\,dx \,+\, (y^2 + \ln x)\,dy = 0$$

Solution.

We have

$$\frac{\partial}{\partial y}(x^3 + \frac{y}{x}) = \frac{1}{x}$$
$$\frac{\partial}{\partial x}(y^2 + \ln x) = \frac{1}{x}.$$

Thus $\frac{\partial}{\partial y}(x^3 + \frac{y}{x}) = \frac{\partial}{\partial x}(y^2 + \ln x)$ and therefore (*) is an exact equation.

We need to find a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = x^3 + \frac{y}{x}$ and $\frac{\partial F}{\partial y} = y^2 + \ln x$. From $\frac{\partial F}{\partial x} = x^3 + \frac{y}{x}$ we get

$$F(x, y) = \int (x^3 + \frac{y}{x})\,dx$$
$$F(x, y) = \frac{x^4}{4} + y\ln x + g(y)$$

Where $g = g(y)$ is some function depending on y only. Now using $\frac{\partial F}{\partial y} = y^2 + \ln x$ we get:

$$\ln x + g'(y) = y^2 + \ln x$$
$$g'(y) = y^2 \quad \text{and hence}$$
$$g(y) = \frac{y^3}{3}.$$

Thus the function $F(x, y) = \frac{x^4}{4} + y \ln x + \frac{y^3}{3}$ satisfies the requirements that $\frac{\partial F}{\partial x} = x^3 + \frac{y}{x}$ and $\frac{\partial F}{\partial y} = y^2 + \ln x$.

Therefore the general solution of (*) is given implicitly by

$$\frac{x^4}{4} + y \ln x + \frac{y^3}{3} = C,$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Problem 5. [20 points]

Solve the following equation on the interval $x > 0$:

$$2xy' - 5y = 6x^3y^4.$$

Solution. Dividing the equation by $2x$ we get

$$y' - \frac{5}{2x}y = 3x^2y^4.$$

This is a Bernoulli equation with $n = 4$. We use the substitution $v = y^{1-4} = y^{-3}$. Hence $y = v^{-1/3}$ and $\frac{dy}{dx} = -\frac{1}{3}v^{-4/3}\frac{dv}{dx}$. Substituting this in the above equation we get

$$-\frac{1}{3}v^{-4/3}\frac{dv}{dx} - \frac{5}{2x}v^{-1/3} = 3x^2v^{-4/3}$$

multiply the equation by $-3v^{4/3}$ to get:

$$\frac{dv}{dx} + \frac{15}{2x}v = -9x^2$$

This is a linear 1-st order equation. We then compute the integrating factor:

$$\rho(x) = e^{\int \frac{15}{2x}\,dx} = e^{\frac{15}{2}\ln x} = x^{15/2}.$$

Multiplying the last equation by $x^{15/2}$ we get:

$$x^{15/2}\frac{dv}{dx} + \frac{15}{2}x^{13/2} = -9x^{19/2}$$
$$\frac{d}{dx}(x^{15/2}v) = 9x^{19/2}$$
$$x^{15/2}v = -\int 9x^{19/2}\,dx$$
$$x^{15/2}v = -9\cdot\frac{2}{21}x^{21/2} + C = -\frac{6}{7}x^{21/2} + C$$
$$v = -\frac{6}{7}x^{3} + Cx^{-15/2}$$

Now using $y = v^{-1/3}$ we conclude that the general solution of the original equation on the interval $x > 0$ is

$$y = (-\frac{6}{7}x^{3} + Cx^{-15/2})^{-1/3}$$

where $C \in \mathbb{R}$ is an arbitrary constant.