Math 385 Midterm Exam 2 (SOLUTIONS)

Prof. I.Kapovich November 11, 2005

Problem 1. [20 points] In this problem you need to mark each option as either TRUE or FALSE. No explanations of your answers are necessary.

(a) [5 points] Let $f(t)$ be a 2π-periodic function such that $f(t) = t^3 + 2t$, where $-\pi < x < \pi$. Let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

(1) [1 point] We have $b_n = 0$ for all $n \geq 1$. [FALSE]
(2) [2 points] There is some $n \geq 0$ such that $a_n \neq 0$. [FALSE]
(3) [2 points] We have $e^{a_0 - 2a_5} = 1$. [TRUE]

(b) [6 points]
Consider the equation

$$(*) \qquad y'' + 3y' + 2y = x \cos x + 3xe^{-x}$$

(1) [2 points] The equation (*) has general solution

$$y = c_1 e^{-x} + c_2 e^{-2x},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.
[FALSE]

(2) The form of y_p according to the Method of Undetermined Coefficients is

$$y_p = (Ax + B) \cos x + (A_1 x + B_1) e^{-x}.$$

[FALSE]

(3) [2 points]
The form of y_p according to the Method of Undetermined Coefficients is

$$y_p = (Ax + B) \cos x + x(A_1 x + B_1) e^{-x}.$$

[FALSE]

Note: the correct form for y_p is:

$$y_p = (Ax + B) \cos x + (A'x + B') \sin x + x(A_1 x + B_1) e^{-x}.$$

(c)[9 points]

Let $f(t)$ be a 2π-periodic function such that

$$f(t) = \begin{cases} t^2+3, & 0<t<\pi \\ -t+1, & -\pi<t<0. \end{cases}$$

Let $f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$ be the Fourier Series of $f(t)$.

(1) [4 points] We have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = 2$$

[TRUE] Use the Convergence Theorem for $t = 0$.

(2) [4 points] We have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n - b_n \sin n) = 2$$

[TRUE] Use the Convergence Theorem for $t = 1$.

(3) [1 point] The series

$$\sum_{n=1}^{\infty}(a_n \cos 9n + b_n \sin 9n)$$

diverges. [FALSE] The Convergence Theorem for $t = 9$ implies that the series converges.

Problem 2. [20 points] Consider the following eigenvalue problem.

$$(\dagger) \qquad y'' + \lambda y = 0, y'(0) = y'(3) = 0.$$

Find all those eigenvalues that are ≥ 0 and find their corresponding eigenfunctions.

Solution.

First, we check if $\lambda = 0$ is an eigenvalue. For $\lambda = 0$ the main equation $y''+\lambda y = 0$ takes the form $y'' = 0$ and its has general solution $y = Ax + B$ where $A, B \in \mathbb{R}$ are arbitrary constants.

Then $y' = A$ and the condition $y'(0) = y'(3) = 0$ yields $A = 0$ and no restrictions on B. Thus for $\lambda = 0$ the system ($\dagger$) has the solution $y = B$ where $B \in \mathbb{R}$ is arbitrary. Therefore $\lambda = 0$ is an eigenvalue with an eigenfunction $y = 1$.

Now let $\lambda > 0$. There is a unique $\alpha > 0$ such that $\lambda = \alpha^2$. The main equation becomes $y'' + \alpha^2 y = 0$ and its general solution is $y = A\cos\alpha x + B\sin\alpha x$, where $A, B \in \mathbb{R}$ are arbitrary constants. Hence $y' = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$. From $y'(0) = 0$ we get $B\alpha \cdot 1 = 0$

and therefore $B = 0$. Thus $y = A \cos \alpha x$ and $y' = -A\alpha \sin \alpha x$. The condition $y'(3) = 0$ gives us

$$-A\alpha \sin 3\alpha = 0$$

If $\sin 3\alpha \neq 0$ then $A = B = 0$, and therefore $y = 0$, so that $\lambda = \alpha^2$ is not an eigenvalue.

If $\sin 3\alpha = 0$ then A is arbitrary and (†) has the solution $y = A \cos \alpha x$ where $A \in \mathbb{R}$ is an arbitrary constant.

We have $\sin 3\alpha = 0$ if and only if $3\alpha = \pi n$ for some $n \in \mathbb{Z}$ (where we must have $n > 0$ since by assumption $\alpha > 0$) and hence $\alpha = \frac{\pi n}{3}$. This gives us eigenvalues $\lambda_n = \frac{\pi^2 n^2}{9}$ with corresponding eigenfunctions $y_n = \cos \frac{\pi n x}{3}$, where $n = 1, 2, 3 \ldots$.

Combining this information with the case $\lambda = 0$ considered earlier, we conclude that the eigenavlue problem (†) has nonnegative eigenvalues $\lambda_n = \frac{\pi^2 n^2}{9}$ with corresponding eigenfunctions $y_n = \cos \frac{\pi n x}{3}$, where $n = 0, 1, 2, 3, \ldots$.

Problem 3. [20 points] Let $f(t)$ be a 2-periodic function such that the Fourier series of $f(t)$ is

$$f(t) = \sum_{n \text{ odd}} \frac{1}{n^2 - 2n^4} \sin \pi n t.$$

Find a formal trigonometric series solution of the following problem:

$$\begin{cases} x'' - 5x = f(t), & 0 < x < 1 \\ x(0) = x(1) = 0. \end{cases}$$

Solution.

We will look for the solution of this system in the form

$$x = \sum_{n=1}^{\infty} b_n \sin \pi n t.$$

Note that the condition $x(0) = x(1) = 0$ is clearly satisfied by any such expression. By termwise differentiation we get

$$x' = \sum_{n=1}^{\infty} \pi n b_n \cos \pi n t$$

$$x'' = \sum_{n=1}^{\infty} -\pi^2 n^2 b_n \sin \pi n t$$

Substituting these formulas for x'' and x into the main equation of the system, we get

$$\sum_{n=1}^{\infty} -\pi^2 n^2 b_n \sin \pi n t - 5 \sum_{n=1}^{\infty} b_n \sin \pi n t = \sum_{n \text{ odd}} \frac{1}{n^2 - 2n^4} \sin \pi n t$$

$$\sum_{n=1}^{\infty} (-\pi^2 n^2 - 5) b_n \sin \pi n t = \sum_{n \text{ odd}} \frac{1}{n^2 - 2n^4} \sin \pi n t$$

By equating the coefficients at the terms $\cos \pi n t$ on the right and the left hand sides, we get

$$b_n = \begin{cases} \frac{1}{(2n^4 - n^2)(\pi^2 n^2 + 5)}, & n \text{ odd} \\ 0, \qquad n \text{ even.} \end{cases}$$

Therefore

$$x = \sum_{n \text{ odd}} \frac{1}{(2n^4 - n^2)(\pi^2 n^2 + 5)} \sin \pi n t.$$

Problem 4. [20 points]

Find a particular solution of the following equation on the interval $-\infty < x < \infty$:

$$y'' + 4y' + 4y = xe^{3x}$$

GIVE ALL THE DETAILS OF YOUR WORK.

Solution.

The complimentary homogeneous equation $y'' + 4y' + 4y = 0$ has characteristic equation $r^2 + 4r + 4 = (r + 2)^2 = 0$. Therefore the complimentary solution is

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Therefore by the Method of Undetermined Coefficients we can look for a particular solution y_p of the form

$$y_p = (Ax + B)e^{3x} = Axe^{3x} + Be^{3x}.$$

Note that there is no duplication in this case. Differentiating the above formula we get

$$y_p' = Ae^{3x} + 3Axe^{3x} + 3Be^{3x} = (A + 3B)e^{3x} + 3Axe^{3x}$$

$$y_p'' = 3(A + 3B)e^{3x} + 3Ae^{3x} + 9Axe^{3x} = (6A + 9B)e^{3x} + 9Axe^{3x}.$$

Substituting these formulas for y_p, y_p', y_p'' into the main equation we get

$$(6A+9B)e^{3x}+9Axe^{3x}+4(A+3B)e^{3x}+12Axe^{3x}+4Axe^{3x}+4Be^{3x}=xe^{3x}$$
$$(6A+9B+4A+12B+4B)e^{3x}+(9A+12A+4A)xe^{3x}=xe^{3x}$$
$$(10A+25B)e^{3x}+25Axe^{3x}=xe^{3x}$$

From here $25A=1$, $10A+25B=0$ and therefore $A=\frac{1}{25}$ and $B=-\frac{2}{125}$.

Thus

$$y_p=\frac{1}{25}xe^{3x}-\frac{2}{125}e^{3x}.$$

Problem 5. [20 points]

Find the Fourier Sine Series of the following function

$$f(t)=2t, \quad 0<t<3.$$

Solution.

We have $L=3$ and for $n\geq 1$

$$b_n=\frac{2}{3}\int_0^3 2t\sin\frac{\pi nt}{3}\,dt= \quad [\text{via } u=2t, dv=\sin\frac{\pi nt}{3}\,dt, v=-\frac{3}{\pi n}\cos\frac{\pi nt}{3}]$$
$$=\Big[-\frac{4t}{\pi n}\cos\frac{\pi nt}{3}\Big]_0^3+\int_0^3\frac{4}{\pi n}\cos\frac{\pi nt}{3}\,dt=-\frac{12}{\pi n}\cos\pi n+\frac{12}{\pi^2n^2}\Big[\sin\frac{\pi nt}{3}\Big]_0^3$$
$$=-\frac{12}{\pi n}\cos\pi n=\frac{12\cdot(-1)^{n+1}}{\pi n}.$$

Therefore the Fourier Sine Series of $f(t)$ is

$$\sum_{n=1}^{\infty}\frac{12\cdot(-1)^{n+1}}{\pi n}\sin\frac{\pi nt}{3}.$$