Some Review Problems

IMPORTANT NOTE: The final exam will be *cumulative*. The Review Problems below are oriented mostly on the last section of the course, but this is ONLY because much of this material was not covered by two midterm exams. You need to go over the material from the first part of the course also, since it will be definitely be included in the final exam.

Problem 1. Show that the functions

$$f(x) = \begin{cases} 1 & x \text{ rational}, \\ -1, & x \text{ irrational} \end{cases}$$

and $g(x) \equiv 3$ are linearly independent on the interval $(-\infty, \infty)$.

Problem 2.

Find all the separated solutions of the system:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi, \\ u(0, y) = u(\pi, y) = 0, & 0 < y < \pi \\ u(x, \pi) = 0, & 0 < x < \pi. \end{cases}$$

Problem 3.

Solve the system:

$$\begin{cases} y_{tt} = 25y_{xx}, & 0 < x < 3, t > 0, \\ y(0, t) = y(3, t) = 0, & t > 0 \\ y(x, 0) = 0, & 0 < x < 3 \\ y_t(x, 0) = 10 \sin 2\pi x, & 0 < x < 3. \end{cases}$$

Problem 4.

An object of mass $m = 1kg$ is attached to a spring with Hooke's constant $k = 4N/m$ and is acted on by a 2π-periodic force $F(t)$ Newtons where $F(t) = 1$ for $0 < t < \pi$ and $F(t) = -1$ for $-\pi < t < 0$.

Determine whether or not pure resonance occurs.

Problem 5.

Let $f(x)$ be a 6-periodic function such that $f(x) = x^2 - x$ for $-3 < x < 3$. Let a_n, b_n be the general Fourier series coefficients of $f(x)$. Find

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{46\pi n}{3} + b_n \sin \frac{46\pi n}{3})$$

and

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \pi n + b_n \sin \pi n).$$

Problem 6.
Find the Fourier Sine Series of the function $f(x)$ defined on the interval $0 < x < 3$ as

$$f(x) = \begin{cases} 1 & 0 < x \leq 2, \\ 5, & 2 < x < 3. \end{cases}$$

Problem 7.
Find the general solution of the following equation:

$$y'' - 6y' + 9y = x^2 e^{3x} + \cos x.$$

Note: This is a pretty long problem!

Problem 8.
Let $y(x,t)$ be the solution of the system:

$$\begin{cases} y_{tt} = 4y_{xx}, & 0 < x < 1, t > 0, \\ y(0,t) = y(1,t) = 0, & t > 0 \\ y(x,0) = x^2, & 0 < x < 1 \\ y_t(x,0) = 0, & 0 < x < 1. \end{cases}$$

Using d'Alambert's solution find the precise value of $y(1/2, 10)$.
Then find the formal infinite series solution of this system.

Problem 9.
Solve the following Dirichlet Problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, 0 < y < b \\ u(x,0) = u(x,b) = u(0,y) = 0, & 0 < x < a, 0 < y < b \\ u(a,y) = g(y), & 0 < y < b \end{cases}$$

Problem 9.
Solve the following Dirichlet Problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < a, y > 0 \\ u(0, y) = u(a, y) = 0, \quad 0 < x < a, 0 < y < b \\ u(x, y) \text{ bounded as } y \to \infty, \\ u(x, 0) = f(x), \quad 0 < x < a \end{cases}$$

Problem 10.

What does it mean for the functions $f(x), g(x)$ to be *orthogonal* on an interval $[a, b]$?

For each pair of the functions below determine if they are orthogonal on the interval indicated:

(1) the functions $f(x) = \cos(3\pi x), g(x) = \sin(4\pi x)$, on the interval $[-\pi/2, \pi/2]$;
(2) the functions $f(x) = \cos(3\pi x), g(x) = \sin(4\pi x)$, on the interval $[-4\pi, 4\pi]$;
(3) the function $f(x) = \sin(x)$ and the function

$$g(x) = \begin{cases} \sin(x), \quad 0 \le x \le \pi, \\ 0, \quad -\pi \le x < 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Problem 11 Consider the equation $y''' + 9y' = x \sin(3x)$.

Then according to the Method of Undetermined Coefficients a particular solution y_p can be found in the form:

(a) $y_p = x(A \sin(3x) + B \cos(3x))$;
(b) $y_p = x^2(A \sin(3x) + B \cos(3x))$;
(c) $y_p = x^2(Ax + B) \sin(3x)$;
(d) None of the above.

Answer

Problem 12 Let $f(t) = t + 5$ for $0 < t < \pi$ and let b_n be the Fourier Sine Series Coefficients for $f(t)$. Then:

(a) We have $1 = \sum_{n=1}^{\infty} nb_n \cos(nt)$ for each $t \in (-\infty, \infty)$.
(b) We have $1 = \sum_{n=1}^{\infty} nb_n \cos(nt)$ for each $t \in (0, \pi)$.
(c) We have $6 \neq \sum_{n=1}^{\infty} b_n \sin(n)$.
(d) None of the above.

Answer:

Problem 13.

Let $f(t) = 3t^2 - t^4$ for $0 < t < 2$. Let a_n, $n = 0, 1, 2, \ldots$, be the coefficients of the Fourier Cosine Series of $f(t)$.

Find the precise values of the following expressions:

(1) $\sum_{n=1}^{\infty} a_n$;
(2) $\sum_{n=1}^{\infty} a_n \cos(\frac{5\pi n}{2})$;
(3) $\sum_{n=1}^{\infty} \frac{\pi n}{2} a_n \sin(\frac{\pi n}{2})$.

Problem 14. Let $f(t)$ be a 6-periodic function defined as

$$f(t) = \begin{cases} t^2, & \text{if } 0 < t < 3, \\ 3t^2, & \text{if } 3 < t < 6, \\ 10, & \text{if } t = 0, 3, 6. \end{cases}$$

Let a_n, b_n be the general Fourier series coefficients for $f(t)$. Then:

(a) We have $b_n = 0$ for all $n \geq 1$.
(b) We have $a_n = 0$ for all $n \geq 0$.
(c) We have $0 = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n$.
(d) None of the above.

Answer:

Problem 15.

Solve the following wave equation problem:

$$\begin{cases} y_t = 25 y_{xx}, & 0 < x < 3, t > 0, \\ y(0,t) = y(3,t) = 0, & t > 0 \\ y(x,0) = 4, & 0 < x < 3 \end{cases}$$

Problem 16

An object of mass $m = 2kg$ is attached to a spring with a spring constant $k = 18\,N/m$ and is acted upon by an odd periodic force $F(t)$ Newtons.

In each of the following cases determine whether or not pure resonance occurs:

(1) The function $F(t)$ has a period of 2π seconds and is defined as:

$$F(t) = \begin{cases} 8, & 0 < t < \pi, \\ -8, & -\pi < t < 0. \end{cases}$$

(2) The function $F(t)$ has a period of 2 seconds and is defined as:

$$F(t) = \begin{cases} 8, & 0 < t < 1, \\ -8, & -1 < t < 0. \end{cases}$$

(3) The function $F(t)$ has a period of 2π seconds and has the Fourier Series:

$$F(t) = \sum_{n \text{ even}} \frac{(-1)^n}{n^4} \sin nt.$$

Problem 17 Select the correct answer for each of the following questions. Each question has exactly one correct answer.

(1) For the initial value problem $\frac{dy}{dx} = \sin(x^2 + y^4)$, $y(1) = 4$ on the interval $I = (-\infty, \infty)$

(a) A solution is guaranteed to exist on I.
(b) A solution is guaranteed to exist on an interval $(1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$.
(c) A unique solution is guaranteed to exist on the interval I.
(d) There are infinitely many solutions on I.

Answer:

(2) The differential equation $y'' - 2xy' + x^3y = 0$ on the interval $I = (-\infty, \infty)$

(a) has exactly one solution;
(b) has the property that for any solutions y_1, y_2 the function $y_1 + 5y_2$ is also a solution;
(c) has characteristic equation $r^2 - 2r + 1 = 0$;
(d) none of the above.

Answer:

(3) Making the substitution $v = y/x$ in an equation $y' = F(y/x)$ transforms this differential equation into

(a) a linear first order differential equation;
(b) a separable differential equation;
(c) an exact equation;
(d) a homogeneous equation;

Answer:

(4) The differential equation $(2x + y^2)dx + (2y + x^2)dy = 0$ is

(a) linear;
(b) homogeneous;
(c) exact;
(d) none of the above.

Answer:

Problem 18

Find a nonzero homogeneous linear equation with constant coefficients such that the functions $y = xe^{2x}$ and $y = xe^{x}\cos(3x)$ both satisfy this equation on $(-\infty, \infty)$.

Problem 19

Give an example of two continuous functions $y_1(x)$ and $y_2(x)$ such that all of the following conditions hold:

(1) The functions y_1, y_2 are linearly dependent on $(-\infty, \infty)$
(2) The Wronskian $W(y_1, y_2)(x) = 0$ for every $x \neq 0, x \in \mathbb{R}$.
(3) The Wronskian $W(y_1, y_2)(0)$ does not exist.
(4) For every $x \in \mathbb{R}$ we have $y_1(x) > 0$ and $y_2(x) > 0$.

Problem 20**

Give an example of two functions $y_1(x)$ and $y_2(x)$ such that all of the following conditions hold:

(1) The functions y_1, y_2 are linearly independent on $(-\infty, \infty)$
(2) The Wronskian $W(y_1, y_2)(x) = 0$ for every $x \in \mathbb{R}$.

Note. This is a pretty hard problem and the functions y_1, y_2 have to be artificially constructed.

Problem 21.

Let $f(t)$ be a 2π-periodic function such that

$$f(t) = \begin{cases} 2t^7 \sin(20t), & \text{if } -\pi < t < 0, 0 < t < \pi, \\ 100, & \text{if } t = 0, \pm\pi. \end{cases}$$

Let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be the Fourier series of f.

For each of the following statements indicate if it is True or False.

(1) We have $a_n = 0$ for all $n \geq 0$.

(2) We have $b_n = 0$ for all $n \geq 1$.

(3) We have $50 = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n$.

(4) We have $2 \cdot 10^7 \sin(200) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(10n) + b_n \sin(10n))$.

(5) We have $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(t)\cos(nt)\,dt$.

Problem 22.

Let $f(t)$ be defined on $[0,5]$ as:

$$f(t) = \begin{cases} t^2 + 10t - 1, & \text{if } 0 < t < 2, \\ 100, & \text{if } t = 0, 2 \\ t + 1, & \text{if } 2 < t < 5 \end{cases}$$

Let b_n be the Fourier Sine Series coefficients of $f(t)$.

Compute the *exact explicit values* (not infinite sum expressions!) of the following:

(1) $\sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{5}$
(2) $\sum_{n=1}^{\infty} b_n \sin \frac{7\pi n}{5}$.

Problem 23. Let $f(t)$ be a 6-periodic function such that the Fourier series of $f(t)$ is

$$f(t) = \sum_{n \text{ even}} \frac{1}{n^2 + 5n^3} \sin \frac{\pi n t}{3}.$$

Find a formal trigonometric series solution of the following problem:

$$\begin{cases} x'' - 4x = f(t), & 0 < x < 3 \\ x(0) = x(3) = 0. \end{cases}$$

Problem 24.

Solve the following Dirichlet problem for the disk R centered at the origin and of radius 2:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \\ u(r,\theta) = u(r, \theta + 2\pi), \\ u(2,\theta) = \sin(3\theta). \end{cases}$$

Some useful Trigonometric and Hyperbolic Trigonometric formulas

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$
$$\sin^2 A = \frac{1-\cos 2A}{2}, \quad \cos^2 A = \frac{1+\cos 2A}{2}.$$

$$\cosh(x) = \frac{e^x+e^{-x}}{2}, \quad \sinh(x) = \frac{e^x-e^{-x}}{2}$$
$$\cosh(-x) = \cosh(x), \quad \sinh(-x) = -\sinh(x),$$
$$\cosh(0) = 1, \quad \sinh(0) = 0,$$
$$\cosh^2 x - \sinh^2 x = 1,$$
$$\cosh'(x) = \sinh(x), \quad \sinh'(x) = \cosh'(x),$$
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$
$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y).$$