

H/wk 4 (Selected Solutions)

2.20 Let $\sigma \in S_n$ fix some j , where $1 \leq j \leq n$. Put $X = \{i \in \mathbb{Z} | 1 \leq i \leq n, i \neq j\}$. Define $\sigma' \in S_X \cong S_{n-1}$ by $\sigma'(i) = \sigma(i)$ for every $i \in X$.

Prove that $\text{sgn}(\sigma) = \text{sgn}(\sigma')$.

Solution.

Let $\sigma = \beta_1 \dots \beta_t$ be a complete factorization of σ . Then $\sigma' = \beta_1 \dots \beta_t(j)$ is a complete factorization of σ' .

Therefore by definition of signum

$$\text{sgn}(\sigma) = (-1)^{n-(t+1)} \text{ and } \text{sgn}(\sigma') = (-1)^{(n-1)-t}.$$

Since $n - (t + 1) = (n - 1) - t$, it follows that $\text{sgn}(\sigma) = \text{sgn}(\sigma')$.

2.24

A permutation $\tau \in S_n$ is called an *adjacency* if it has the form $\tau = (i \ i+1)$ for some $i < n$.

Prove that if i, j and $(i \ j) \in S_n$ then $(i \ j)$ can be represented as a product of an odd number of adjacencies.

Solution.

We will prove this by induction on $m = j - i$.

1) Base of Induction. Suppose that $m = j - i = 1$. Then $j = i + 1$ and $(i \ j) = (i \ i + 1)$ is an adjacency, so that the statement holds.

2) Inductive Step. Let $m \geq 1$ and suppose that the statement has been proved for m . Suppose $j - i = m + 1$. Then $(j - 1) - i = m$ and by the inductive hypothesis there exists a representation

$$(i \ j - 1) = \tau_1 \dots \tau_k$$

where τ_s are adjacencies and k is odd.

We know that if $\alpha \in S_n$ is such that $\alpha(i) = i, \alpha(j - 1) = j$ then $\alpha(i \ j - 1)\alpha^{-1} = (i \ j)$. Take $\alpha = (j - 1 \ j)$. Then $\alpha = \alpha^{-1}$ is an adjacency and $\alpha(i) = i, \alpha(j - 1) = j$. Therefore $(j - 1 \ j)(i \ j - 1)(j - 1 \ j) = (i \ j)$ and hence

$$(j - 1 \ j)\tau_1 \dots \tau_k(j - 1 \ j) = (i \ j).$$

Thus $(i \ j)$ can be represented as a product of $k + 2$ adjacencies (which is an odd number since k was odd), which completes the inductive step.