H/wk 5 (Selected Solutions)

2.33(ii)

How many elements of order two are there in S_n?

Solution.

An element of S_n has order two if and only if this element is a product of k disjoint transpositions, where $1 \le k \le n/2$.

Fix an integer k such that $1 \le k \le n/2$. The product of k disjoint transpositions has the form

$$\alpha = (i_1\ j_1)(i_2\ j_2)\dots(i_k\ j_k)$$

where the numbers $i_1, j_1, i_2, j_2, \dots, i_k, j_k$ are distinct elements of the set $\{1, \dots, n\}$.

Since $(i\ j) = (j\ i)$, the number of ways to choose the first transposition $(i_1\ j_1)$ is equal to the number of ways to choose a 2-element subset from a set with n elements, that is to $\binom{n}{2}$. For each of these choices the number of ways to choose the second transposition $(i_2\ j_2)$ is $\binom{n-2}{2}$. For each of these choices the number of ways to choose the third transposition is $\binom{n-4}{2}$. And so on.

This results in $\prod_{s=0}^{k-1} \binom{n-s}{2}$ ways to choose an ordered product of k disjoint transpositions. However, disjoint permutations commute and thus each reshuffling of such a product gives us the same permutation in S_n. Therefore the number of elements of S_n that are products of k disjoint transpositions is:

$$\frac{1}{k!} \prod_{s=0}^{k-1} \binom{n-s}{2}.$$

Hence the total number of elements of order two in S_n is:

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k!} \prod_{s=0}^{k-1} \binom{n-s}{2}.$$

2.39 Prove that if G is a finite group where every element has a square root, then every element in G has a unique square root.

Solution.

Consider the function $f : G \to G$ defines as $f(x) = x^2$ for every $x \in G$. By assumption this function f is "onto". Since G is a finite set, this implies that f is injective. Therefore every element in G has a unique square root, as required.