Homework 9 (selected solutions)

2.104

Prove that no pair of the following groups of order 8 are isomorphic (I am omitting the group D_8 since it was not discussed in class):

$$\mathbb{I}_8, \mathbb{I}_4 \times \mathbb{I}_2, I_2 \times I_2 \times I_2, \mathbf{Q}.$$

Solution.

The group $\mathbf{Q}$ is non-abelian while the groups $\mathbb{I}_8, \mathbb{I}_4 \times \mathbb{I}_2, I_2 \times I_2 \times I_2$ are abelain. Therefore $\mathbf{Q}$ is not isomorphic to any of the groups $\mathbb{I}_8, \mathbb{I}_4 \times \mathbb{I}_2, I_2 \times I_2 \times I_2$.

The group $\mathbb{I}_8$ has an element of order 8 while the groups $\mathbb{I}_4 \times \mathbb{I}_2, I_2 \times I_2 \times I_2$ do not have elements of order 8. Therefore $\mathbb{I}_8$ is not isomorphic to either of the groups $\mathbb{I}_4 \times \mathbb{I}_2, I_2 \times I_2 \times I_2$.

The group $\mathbb{I}_4 \times \mathbb{I}_2$ has an element of order 4 while the group $I_2 \times I_2 \times I_2$ has no elements of order 4. Therefore $\mathbb{I}_4 \times \mathbb{I}_2 \not\cong I_2 \times I_2 \times I_2$.

2.112 (i) How many permutations in S_5 commute with $(1\ 2)(3\ 4)$ and how many even permutations commute with $(1\ 2)(3\ 4)$?

(ii) How many permutations in S_7 commute with $(1\ 2)(3\ 4\ 5)$?

(iii) Exhibit all the permutations in S_7 that commute with $(1\ 2)(3\ 4\ 5)$.

Solution.

First we need to prove:

Lemma Let $H \leq G$ be a subgroup of index 2. Let $A \leq G$. Then $[A : A \cap H] \leq 2$.

Proof. If $A \subseteq H$ then $A = A \cap H$ and $[A : A \cap H] = 1$. Suppose now that $A \not\subseteq H$. Let $a \in A - H$. Then $G = H \cup aH$. We claim that $A = (A \cap H) \cup a(A \cap H)$. This would imply that $[A : A \cap H] = 2$.

Let $b \in A$ be arbitrary. If $b \in H$ then $b \in A \cap H$. Suppose now $b \notin H$ then $b \in aH$ and $b = ah$ for some $h \in H$. Therefore $h = a^{-1}b \in A$ since $a, b \in A$. This $h \in A \cap H$ and $b \in a(A \cap H)$. Since $b \in H$ was arbitrary, we have proved that $A = (A \cap H) \cup a(A \cap H)$, as required. □

(i) The set of permutations commuting with $x = (1\ 2)(3\ 4)$ in S_5 is the centralizer $C_{S_5}(x)$. Therefore $[S_5 : C_{S_5}(x)] = |x^{S_5}|$. The conjugacy class x^{S_5} consists of all permutations in S_5 with the cycle structures of being the product of two disjoint cycles. Therefore

$$|x^{S_5}| = \binom{5}{2}\binom{3}{2}\frac{1}{2} = 15.$$

Therefore

$$|C_{S_5}(x)| = \frac{|S_5|}{[S_5 : C_{S_5}(x)]} = \frac{5!}{15} = 8.$$

By the Lemma above $[C_{S_5}(x) : C_{S_5}(x) \cap A_5] \leq 2$ since $[S_5 : A_5] = 2$. The permutation $(1\ 2)$ is odd but it commutes with x. Thus $(1\ 2) \in C_{S_5}(x) - A_5$

and therefore $[C_{S_5}(x) : C_{S_5}(x) \cap A_5] \neq 1$. Hence $[C_{S_5}(x) : C_{S_5}(x) \cap A_5] = 2$ and so

$$|C_{S_5}(x) \cap A_5| = \frac{1}{2}\,|C_{S_5}(x)| = 8/2 = 4.$$

Thus there are 4 even permutations in S_5 that commute with x.

(ii) Denote $y = (1\ 2)(3\ 4\ 5) \in S_7$. By the same argument as above

$$[S_7 : C_{S_7}(y)] = |y^{S_7}|.$$

The conjugacy class y^{S_7} consists of all permutations in S_7 with the cycle structure of the disjoint product of a 2-cycle and a 3-cycle. The number of such permutations is:

$$|y^{S_7}| = \binom{7}{2} 5 \cdot 4 \cdot 3 \frac{1}{3} = 420.$$

Therefore the number of elements of S_7 that commute with y is

$$|C_{S_7}(y)| = \frac{|S_7|}{[S_7 : C_{S_7}(y)]} = \frac{7!}{420} = 12.$$

(iii) We know that for any $\alpha \in S_7$

$$\alpha y \alpha^{-1} = (\alpha(1)\ \alpha(2))(\alpha(3)\ \alpha(4)\ \alpha(5)).$$

Thus $\alpha y \alpha^{-1} = y$ if and only if $(\alpha(1)\ \alpha(2)) = (1\ 2)$ and $(\alpha(3)\ \alpha(4)\ \alpha(5)) = (3\ 4\ 5)$. For each such α either $\alpha(6) = 6, \alpha(7) = 7$ or $\alpha(6) = 7, \alpha(7) = 6$.

Thus α commutes with y if and only if

$$(\alpha(1), \alpha(2)) \in \{(1,2),(2,1)\} \text{ and}$$
$$(\alpha(3), \alpha(4), \alpha(5)) \in \{(3,4,5),(4,5,3),(5,3,4)\} \text{ and}$$
$$(\alpha(6), \alpha(7)) \in \{(6,7),(7,6)\}.$$

Hence

$$C_{S_7}(y) = \{\begin{pmatrix}1&2&3&4&5&6&7\\1&2&3&4&5&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\1&2&3&4&5&7&6\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4&5&6&7\\1&2&4&5&3&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\1&2&4&5&3&7&6\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4&5&6&7\\1&2&5&3&4&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\1&2&5&3&4&7&6\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4&5&6&7\\2&1&3&4&5&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\2&1&3&4&5&7&6\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4&5&6&7\\2&1&4&5&3&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\2&1&4&5&3&7&6\end{pmatrix},$$
$$\begin{pmatrix}1&2&3&4&5&6&7\\2&1&5&3&4&6&7\end{pmatrix}, \begin{pmatrix}1&2&3&4&5&6&7\\2&1&5&3&4&7&6\end{pmatrix}\}.$$