Quiz 5 (Solutions); Friday, March 4, 2005

For each of the following inclusions determine if the specified subset is a subgroup. If it is not a subgroup, explain why not.

(a) $X \subseteq GL_n(\mathbb{R})$, where

$$X = \{A \in GL_n(\mathbb{R}) : \det(A) > 0\}.$$

(b) $X \subseteq \mathbb{Q}$, where

$$X = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \text{ and } 6|n\},$$

and where $\mathbb{Q}$ is considered as an additive group.

(c) $X \subseteq S_8$, where

$$X = \{\alpha \in S_8 : sgn(\alpha) = 1\}.$$

(d) $X \subseteq \mathbb{C}$ where

$$X = \{a + bi \in \mathbb{C} : a, b \in \mathbb{R} \text{ and at least one of } a, b \text{ is integer}\},$$

and where $\mathbb{C}$ is considered as an additive group.

Solution.

(a) Yes, this is a subgroup.

Indeed, $\det(I_n) = 1 > 0$ and so $I_n \in X$.

If $A, B \in X$ then $\det(A) > 0, \det(B) > 0$. Hence $\det(AB) = \det(A)\det(B) > 0$ and therefore $AB \in X$.

If $A \in X$ then $\det(A) > 0$ and $det(A^{-1}) = \frac{1}{\det(A)} > 0$, so that $A^{-1} \in X$.

(b) Yes, this is a subgroup.

Note that $X = \mathbb{Q}$. Indeed let $r \in \mathbb{Q}$ be arbitrary. Then there exist $a, b \in \mathbb{Z}$, $b \neq 0$ such that $r = \frac{a}{b} = \frac{6a}{6b}$. Therefore $r \in X$.

(c) Yes, it is a subgroup.

(d) No, this is not a subgroup. Indeed, $\frac{1}{2} \in X$ and $\frac{1}{2}i \in X$ but $\frac{1}{2} + \frac{1}{2}i \notin X$.