

Extra Credit Problem Set 2; Due Friday, March 28.

Print your name:

1.

Let G be a group. For $a, b \in G$ define $[a, b] := aba^{-1}b^{-1}$. The element $[a, b] \in G$ is called the *commutator of a and b* .

Let $S = \{[a, b] : a, b \in G\} \subseteq G$. Put $[G, G] = \langle S \rangle \leq G$ to be the subgroup of G generated by S (see Extra Credit Set 1 for definitions).

(a) Prove that the subgroup $[G, G]$ is normal in G .

(b) Prove that the quotient group $G/[G, G]$ is abelian.

(c) Prove that for a homomorphism $\phi : G \rightarrow G_1$ we have

$$[G, G] \leq \text{Kern}(\phi) \iff \phi(G) \text{ is abelian.}$$

The subgroup $[G, G]$ is called the *commutator subgroup of G* and the quotient group $G/[G, G]$ is called the *abelianization of G* .

2. Let X be a nonempty set. Let $X^{-1} = \{x^{-1} | x \in X\}$ be the set of formal inverses of X . Denote $\Sigma = X \cup X^{-1}$. Denote by Σ^* the set of all words in the alphabet Σ (including the empty word, denoted by ϵ), that is:

$$\Sigma^* = \{y_1 \dots y_n | n \geq 0, y_i \in \Sigma, 1 \leq i \leq n\}.$$

We say that a word $w \in \Sigma^*$ is *freely reduced* if it does not contain any subwords of the form xx^{-1} , $x^{-1}x$, where $x \in X$. Let

$$F(X) := \{w \in \Sigma^* : w \text{ is freely reduced}\}.$$

An *elementary free reduction* on a word $w \in \Sigma^*$ consists in removing a subword of the form xx^{-1} or $x^{-1}x$, where $x \in X$. E.g. for $X = \{a, b\}$, the word abb is obtained from $aba^{-1}ab$ by an elementary free reduction.

Define a binary operation \cdot on $F(X)$ as follows. For $u, v \in F(X)$ put $u \cdot v \in F(X)$ to be the word obtained from the concatenation $uv \in \Sigma^*$ by a maximal possible chain of elementary free reductions.

E.g. for $X = \{a, b\}$ we have $aba^{-1}b \cdot b^{-1}baab^{-1} = abaab^{-1}$.

(a) Prove that $(F(X), \cdot)$ is a group. This group is called *the free group on X* .

Hint. The tricky property here is associativity.

Try the following. Prove, by induction on the length of $w \in \Sigma^*$, that for every $w \in \Sigma^*$ the freely reduced word $\bar{w} \in F(X)$, obtained from w by a maximal chain of elementary free reductions, is unique and does not depend on the sequence of elementary free reductions chosen. To see this, prove first that if $w \rightarrow w_1$ and $w \rightarrow w_2$ are elementary free reductions, then there exist $w_3 \in \Sigma^*$, such that $w_1 \rightarrow w_3$ and $w_2 \rightarrow w_3$ are elementary free reductions.

Then use associativity of the concatenation operation in Σ^* to conclude that $(F(X), \cdot)$ is associative.

(b) Prove that for any group G and any function $f : X \rightarrow G$ there exists a unique homomorphism $\phi : F(X) \rightarrow G$ such that $\phi(x) = f(x)$ for every $x \in X$.