Extra Credit Problems Set 1; Due Wednesday, April 22

TO BE TURNED IN SEPARATELY FROM THE REGULAR HOMEWORK

Problem 1.

Prove that the additive groups $(\mathbb{Q}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$ are not isomorphic.

Hint: Show that every finitely generated subgroup of Q is cyclic but that $\mathbb{Q} \times \mathbb{Q}$ has a non-cyclic finitely generated subgroup.

Problem 2.

(a) Let $f : R \to R'$ be a ring isomorphism.

Prove that $a \in R$ is a unit (that is, if a has a multiplicative inverse in R) if and only if $f(a) \in R'$ is a unit.

(b) Find all units in $\mathbb{Z}[x]$.

(c) Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ are not isomorphic.

Hint. Use the results of parts (a) and (b).

Here $\mathbb{Z}[[x]]$ is the *ring of formal power series* with coefficients in $\mathbb{Z}$, that is

$$\mathbb{Z}[[x]] = \{\sum_{i=0}^{\infty} a_n x^n \mid a_i \in \mathbb{Z}\}$$

with the termwise addition and with multiplication defined in a similar way to that of multiplication of ordinary polynomials:

If $f = \sum_{i=0}^{\infty} a_n x^n$ and $g = \sum_{i=0}^{\infty} b_n x^n$ then $fg := \sum_{k=0}^{\infty} c_k x^k$ where for $k = 0, 1, 2, \ldots$ we have

$$c_k = \sum_{i=0}^{k} a_i bk - i = a_0 b_k + a_1 b_{k-1} + \ldots a_{k-1} b_1 + a_k b_0.$$