Extra Credit Problems Set 1; Due Wednesday, April 22

SOLUTIONS

Problem 1.

Prove that the additive groups $(\mathbb{Q},+)$ and $(\mathbb{Q}\times\mathbb{Q},+)$ are not isomorphic.

Hint: Show that every finitely generated subgroup of Q is cyclic but that $\mathbb{Q}\times\mathbb{Q}$ has a non-cyclic finitely generated subgroup.

Solution.

We will first prove that every finitely generated subgroup of Q is cyclic. Indeed, let $X\subseteq\mathbb{Q}$ be a finite subset and $H=\langle Q\rangle$. Then $X=\{\frac{p_1}{q_1},\dots,\frac{p_k}{q_k}\}$ where $p_i,q_i\in\mathbb{Z}$ and $q_i>0$.

Then

$$H=\langle Q\rangle=\{n_1\frac{p_1}{q_1}+\cdots+n_k\frac{p_k}{q_k}|n_i\in\mathbb{Z}\}=$$

$$\{\frac{n_1p_1q_2\dots q_k+\cdots+n_kp_kq_1\dots q_{k-1}}{q_1\dots q_k}|n_i\in\mathbb{Z}\}\subseteq\{\frac{n}{q_1\dots q_k}|n\in\mathbb{Z}\}=\langle\frac{1}{q_1\dots q_k}\rangle$$

Thus H is a subgroup of a cyclic group $\langle\frac{1}{q_1\dots q_k}\rangle$ and hence H is cyclic itself. This proves the claim that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

On the other hand, $(\mathbb{Q}\times\mathbb{Q},+)$ has some finitely generated subgroups that are not cyclic. For example, $\langle(1,0),(0,1)\rangle=\mathbb{Z}\times\mathbb{Z}$ is finitely generated but not cyclic.

This implies that the groups $\mathbb{Q}$ and $\mathbb{Q}\times\mathbb{Q}$ are not isomorphic.

Problem 2.

(a) Let $f:R\to R'$ be a ring isomorphism.

Prove that $a\in R$ is a unit (that is, if a has a multiplicative inverse in R) if and only if $f(a)\in R'$ is a unit.

(b) Find all units in $\mathbb{Z}[x]$.

(c) Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ are not isomorphic.

Hint. Use the results of parts (a) and (b).

Here $\mathbb{Z}[[x]]$ is the *ring of formal power series* with coefficients in $\mathbb{Z}$, that is

$$\mathbb{Z}[[x]]=\{\sum_{i=0}^{\infty}a_nx^n \mid a_i\in\mathbb{Z}\}$$

with the termwise addition and with multiplication defined in a similar way to that of multiplication of ordinary polynomials:

If $f=\sum_{i=0}^{\infty}a_nx^n$ and $g=\sum_{i=0}^{\infty}b_nx^n$ then $fg:=\sum_{k=0}^{\infty}c_kx^k$ where for $k=0,1,2,\dots$ we have

$$c_k=\sum_{i=0}^{k}a_ibk-i=a_0b_k+a_1b_{k-1}+\dots a_{k-1}b_1+a_kb_0.$$

Solution.

(a) This is an easy exercise in definitions and we omit the details.

(b) Note that if $f,g\in\mathbb{Z}[x]$ are nonzero polynomials then $deg(fg)=deg(f)+deg(g)$. Indeed, let $f=a_0+a_1x+\cdots+a_nx^n$ and $g=b_0+b_1x+\cdots+b_mx^m$ where $a_i,b_j\in\mathbb{Z}$ and where $a_n\neq 0,b_m\neq 0$. Then

$$fg=a_0b_0+(a_0b_1+a_1b_0)x+\dots a_nb_mx^{m+n}$$

Since $a_n \neq 0, b_m \neq 0$ and since $\mathbb{Z}$ has no zero divisors, it follows that $a_n b_m \neq 0$, so that $def(f,g) = m + n = deg(f) + deg(g)$.

Suppose now that $f(x) \in \mathbb{Z}[x]$ is a unit and that $g(x) \in \mathbb{Z}[x]$ is such that $fg = 1$ in $\mathbb{Z}[x]$. Then $deg(f)+deg(g) = deg(1) = 0$ which implies that $deg(f) = deg(g) = 0$. Thus $f = a$ and $g = b$ for some $a, b \in \mathbb{Z}$.

Then $fg = 1$ gives us $ab = 1$ in $\mathbb{Z}$ and hence either $a = b = 1$ or $a = b = -1$.

Therefore the set of units in $\mathbb{Z}[x]$ is $\{-1, 1\}$.

(c) Suppose $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ are isomorphic rings and let $\phi : \mathbb{Z}[x] \to \mathbb{Z}[[x]]$ be a ring isomorphism.

By part (a) ϕ maps bijectively the set of units in $\mathbb{Z}[x]$ to the set of units in $\mathbb{Z}[[x]]$. We know that the only units in $\mathbb{Z}[x]$ are $1, -1$ and it is clear that $1, -1$ are also units in $\mathbb{Z}[[x]]$. Thus if we find at least one more unit in $\mathbb{Z}[[x]]$, we will obtain a contradiction.

It is easy to see, by using the definition of multiplication in $\mathbb{Z}[[x]]$, that

$$(1 - x)(1 + x + x^2 + x^3 + \cdots + x^i + \dots) = 1 \quad \text{in } \mathbb{Z}[[x]].$$

Thus $1 - x$ is a unit in $\mathbb{Z}[[x]]$, which gives us a contradiction.