H/wk 12, Solutions to selected problems

Ch. 3.4, Problem 1

In each case determine if the map θ is a ring homomorphism.

(a) $\theta : Z_3 \to \mathbb{Z}_{12}$, $\theta(r) = 4r$.

Answer:

No, this is not a ring homomorphism. Indeed, $\theta([1]_3) = [4]_{12} \neq [1]_{12}$.

(b) $\theta : Z_4 \to \mathbb{Z}_{12}$, $\theta(r) = 3r$.

Answer:

No, this is not a ring homomorphism. Indeed, $\theta([1]_4) = [3]_{12} \neq [1]_{12}$.

(c) $\theta : R \times R \to R$, where $\theta(r, s) = r + s$.

Answer:

No, this is not a ring homomorphism. Indeed, $1_{R\times R} = (1_R, 1_R)$ but

$$\theta(1_{R\times R}) = \theta(1_R, 1_R) = 1_R + 1_R \neq 1_R \text{ when } R \neq 0.$$

(d) $\theta : R \times R \to R$, where $\theta(r, s) = rs$.

Answer:

No, this is not a ring homomorphism.

Indeed, $\theta(1, 0) = \theta(0, 1) = 0$. But $\theta((1, 0) + (0, 1)) = \theta(1, 1) = 1 \neq 0 + 0$.

(e) $\theta : F(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ where $\theta(f) = f(1)$.

Answer:

Yes, this is a ring homomorphism. Indeed, for $f, g \in F(\mathbb{R}, \mathbb{R})$ we have $(fg)(x) = f(x)g(x)$ and $(f + g)(x) = f(x) + g(x)$ for every $x \in \mathbb{R}$. Hence

$$\theta(f + g) = (f + g)(1) = f(1) + g(1) = \theta(f) + \theta(g)$$

and

$$\theta(fg) = (fg)(1) = f(1)g(1) = \theta(f)\theta(g).$$

Also, for the function $\mathbf{1} : \mathbb{R} \to \mathbb{R}$, $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}$ we have $\theta(\mathbf{1}) = \mathbf{1}(1) = 1$. Thus θ is indeed a ring homomorphism.

Ch. 3.4, Problem 5

Let $\theta : R \to R_1$ be an onto ring homomorphism. Show that $\theta(Z(R)) \subseteq Z(R_1)$. Give an example showing that this need not be equality.

Solution.

The proof that $\theta(Z(R)) \subseteq Z(R_1)$ is easy and we omit the details.

For the example where $\theta : R \to R_1$ be an onto ring homomorphism but $\theta(Z(R)) \neq Z(R_1)$ we will use Example 4 from Chapter 3.4. Put $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, $R_1 = \mathbb{Z} \times \mathbb{Z}$ and $\theta : R \to R_1$ given by $\theta(\begin{bmatrix} r & s \\ 0 & t \end{bmatrix}) = (r, t)$ for $r, s, t \in \mathbb{Z}$. It is checked in Example 4 that θ is an onto ring homomorphism. It is also clear that $R_1 = \mathbb{Z} \times \mathbb{Z}$ is a commutative ring so that $Z(R_1) = R_1$.

We claim that $Z(R) = \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{Z}\}$. If this claim is true then $\theta(Z(R)) = \{(a, a) | a \in \mathbb{Z}\} \neq \mathbb{Z} \times \mathbb{Z} = Z(R_1)$, so that this example would have the required properties.

Thus it remains to verify the claim.

Suppose $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in Z(R)$. We have $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & b \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & c \end{bmatrix}$. Since $A \in Z(R)$, thies yields $\begin{bmatrix} 2a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & c \end{bmatrix}$ and hence $b = 0$. Thus $A = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$.

We also have $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & c \\ 0 & c \end{bmatrix}$. Since $A \in Z(R)$, this implies $a = c$, so that $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. It is also easy to see that for every $a \in \mathbb{Z}$ we do have $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in Z(R)$. Thus $Z(R) = \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{Z}\}$, as claimed.

Ch. 3.4, Problem 11

(a) Show that $x^3 - 8x^2 + 5x + 3 = 0$ has no solutions $x \in \mathbb{Z}$.

Solution.

Suppose, on the contrary, that there exists $x \in \mathbb{Z}$ such that $x^3 - 8x^2 + 5x + 3 = 0$. Consider the ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_5$ defined by $\phi(n) = \overline{n}$ where $n \in \mathbb{Z}$.

Then

$$\overline{0} = \phi(0) = \phi(x^3 - 8x^2 + 5x + 3) = \overline{x}^3 - \overline{8}\overline{x}^2 + \overline{3} = \overline{x}^3 + \overline{2}\overline{x}^2 + \overline{3}.$$

Thus $\overline{x}^3 + \overline{2}\overline{x}^2 + \overline{3} = \overline{0}$ in $\mathbb{Z}_5$. By substituting the possible values of $\overline{x} = \overline{0}, \overline{1}, \overline{2}, \overline{-2}, \overline{-1}$ into the left-hand side of the above equation, we get:

$$\begin{aligned}
\overline{0}^3 + \overline{2} \cdot \overline{0}^2 + \overline{3} &= \overline{3} \neq \overline{0} \\
\overline{1}^3 + \overline{2} \cdot \overline{1}^2 + \overline{3} &= \overline{1} \neq \overline{0} \\
\overline{2}^3 + \overline{2} \cdot \overline{2}^2 + \overline{3} &= \overline{4} \neq \overline{0} \\
\overline{(-2)}^3 + \overline{2} \cdot \overline{(-2)}^2 + \overline{3} &= \overline{3} \neq \overline{0} \\
\overline{(-1)}^3 + \overline{2} \cdot \overline{(-1)}^2 + \overline{3} &= \overline{4} \neq \overline{0}.
\end{aligned}$$

Since $\mathbb{Z}_5 = \{overline0, \overline{1}, \overline{2}, \overline{-2}, \overline{-1}\}$, this yields a contradiction with the fact that $\overline{x}^3 + \overline{2}\overline{x}^2 + \overline{3} = \overline{0}$ in $\mathbb{Z}_5$.

Hence there does not exist $x \in \mathbb{Z}$ such that $x^3 - 8x^2 + 5x + 3 = 0$.

Ch. 3.4, Problem 21

Show that there is no ring homomorphism $\mathbb{C} \to \mathbb{R}$.

Solution.

Suppose that there exists a ring homomorphism $f : \mathbb{C} \to \mathbb{R}$. Recall that, by definition of a ring homomorphism, we have $f(1) = 1$. Hence $f(-1) = -1$ since $f : \mathbb{C} \to \mathbb{R}$ is a also group homomorphism between the additive abelian groups of $\mathbb{C}$ and $\mathbb{R}$.

Let $r := f(i) \in \mathbb{R}$. Since $i^2 = -1$ in $\mathbb{C}$ and since f is a ring homomorphism, we have

$$-1 = f(-1) = f(i^2) = f(i)^2 = r^2.$$

Thus $r \in \mathbb{R}$ is a real number such that $r^2 = -1$. This is a contradiction since for every $r \in \mathbb{R}$ we have $r^2 \geq 0$. Hence there does not exist a ring homomorphism $\mathbb{C} \to \mathbb{R}$.

Ch. 3.4, Problem 27

Let S be a ring and let $R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix} = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in S\}$ be the upper triangular matrix ring over S. Show that $A = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \{\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} | s \in S\}$ is an ideal in S and that $R/S \cong S \times S$.

Solution.

Consider the function $f : R \to S \times S$ defined by $f(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) := (a, c)$ for $a, b, c \in S$.

We claim that f is a ring homomorphism.

Indeed,

$$f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{bmatrix}\right) =$$

$$= (a_1 + a_2, c_1 + c_2) = (a_1, c_1) + (a_2, c_2) = f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}\right).$$

Similarly,

$$f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{bmatrix}\right) =$$

$$= (a_1 a_2, c_1 c_2) = (a_1, c_1)(a_2, c_2) = f\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}\right) f\left(\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}\right).$$

Also, $f(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) := (1, 1) = 1_{S \times S}$. Thus f is indeed a ring homomorphism.

Also, by definition of f and A we have $Ker(f) = \{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} | b \in S\} = A$. Note that this already implies that A is an ideal in R.

Finally observe that f is onto. Indeed, let $s, s' \in S$ be arbitrary so that $(s, s') \in S \times S$ is an arbitrary element. Then $\begin{bmatrix} s & 0 \\ 0 & s' \end{bmatrix} \in R$ and $f(\begin{bmatrix} s & 0 \\ 0 & s' \end{bmatrix}) = (s, s')$.

Thus $f : R \to S \times S$ is an onto ring homomorphism with $Ker(f) = A$. Therefore by the Isomorphism Theorem for rings we have $R/A \cong S \times S$ as rings.

Ch. 3.4, Problem 32

Prove the **Second Isomorphism Theorem**: If A is an ideal of R, and S is a subring of R then $S + A = \{s + a | s \in S, a \in A\}$ is a subring of R, and, moreover, A and $S \cap A$ are ideals of $S + A$ and S respectively and $(S + A)/A \cong S/(S \cap A)$.

Solution.

First check that $S + A = \{s + a | s \in S, a \in A\}$ is a subring of R. Indeed, $1 = 1 + 0 \in S + A$. Also, for any $s_1, s_2 \in S$, $a_1, a_2 \in A$ we have $(s_1 + a_1) + (s_2 + a_2) = (s_1 + s_2) + (a_1 + a_2) \in S + A$ since $s_1 + s_2 \in S$ and $a_1 + a_2 \in A$. Similarly, in the above situation we have

$$(s_1 + a_1)(s_2 + a_2) = s_1 s_2 + (a_1 s_2 + s_1 a_2 + a_1 a_2) \in S + A$$

since $s_1 s_2 \in S$ (because S is a subring of R) and $a_1 s_2 + s_1 a_2 + a_1 a_2 \in A$ (because A is an ideal in R). Thus indeed, $S + A$ is an ideal in R and, moreover $A \subseteq S + A$ since for every $a \in A$ we have $a = 0 + a \in S + A$.

Since A is an ideal in R and $A \subseteq S + A \subseteq R$, it follows from the definition of an ideal that A is an ideal in $S + A$.

We now have to show that $(S + A)/A \cong S/(S \cap A)$.

Define $f : S \to (S + A)/A$ by $f(s) = s + A$ for $s \in S$.

We claim that f is a ring homomorphism. Indeed,

$$f(s_1 + s_2) = s_1 + s_2 + A = (s_1 + A) + (s_2 + A) = f(s_1) + f(s_2)$$
$$f(s_1 s_2) = s_1 s_2 + A = (s_1 + A) \cdot (s_2 + A) = f(s_1) f(s_2)$$

Finally, $f(1) = 1 + A$. Thus indeed f is a ring homomorphism.

We claim that f is onto. Indeed, for any $s \in S, a \in A$ we have $f(s) = s + A = s + a + A$, so that f is indeed onto.

We also claim that $Ker(f) = S \cap A$. Indeed for $s \in S$ we have

$$f(s) = 0 + A \iff s + A = 0 + A \iff s \in A \iff s \in S \cap A.$$

Thus $f : S \to (S + A)/A$ is an onto ring homomorphism with $Ker(f) = S \cap A$. Therefore $S \cap A$ is an ideal in S and, by the Isomorphism Theorem, $(S + A)/A \cong S/(S \cap A)$, as required.