H/wk 14, Solutions to selected problems

Ch. 8.3, Problem 13

Let $G = (\mathbb{R}, +)$, and define $a \cdot z = e^{ia}z$ for $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

Show that $\mathbb{C}$ is a G-set, describe the action geometrically and find all orbitz and stabilizers.

Solution.

First, we check that $\mathbb{C}$ is a G-set. For any $z \in \mathbb{C}$ we have $0 \cdot z = e^0 z = z$. Also, for any $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$ we have

$$a \cdot (b \cdot z) = a \cdot (e^{ib}z) = e^{ia}e^{ib}z = e^{i(a+b)}z = (a + b) \cdot z.$$

Thus this is indeed a group action of $G = (\mathbb{R}, +)$ on $\mathbb{C}$.

Recall, that in polar coordinates when two complex numbers are multiplies, their polar angles are added and their absolute values are multiplied. Recall also that $e^{ia} = \cos a + i \sin a$. Hence, geometrically, for $a \in \mathbb{R}$ and $z \in \mathbb{C}$ the point $a \cdot z = e^{ia}z$ is obtained by rotating the point z around the origin counterclockwise by angle a.

Thus for $z_0 \neq 0$ we have $Gz_0 = \{z \in \mathbb{C} : |z| = |z_0|\}$, so that the orbit Gz_0 is the circle around the origin of radius $|z_0|$. For $z_0 = 0 \in \mathbb{C}$ we have $Gz_0 = \{z_0\} = \{0\}$.

For $z_0 \in \mathbb{C}$, $z_0 \neq 0$ the stabilizer of z_0 in G is

$$Stab_G(z_0) = \{2\pi n : n \in \mathbb{Z}\}.$$

Finally, for $z_0 = 0 \in \mathbb{C}$ we have $Stab_G(z_0) = G$.

Ch. 8.3, Problem 23

Let X be a G-set and let $x, y \in X$.

(a) Show that the stabilizer $S(x)$ is a subgroup of G.

Solution.

Note that by definition of a group action $e \cdot x = x$, so that $e \in S(x)$.

Let $g, h \in S(x)$, that is $g \cdot x = x$ and $h \cdot x = x$.

Then

$$(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$$

and hence $gh \in S(x)$.

Finally, let $g \in S(x)$, so that $g \cdot x = x$.

Then

$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x$$

so that $g^{-1} \in S(x)$. Thus indeed $S(x) \leq G$ is a subgroup of G.

(b) If $x \in X$ and $g \in G$, show that $S(b \cdot x) = bS(x)b^{-1}$.

Solution.

Let $g \in S(b \cdot x)$ be arbitrary. Thus $g \cdot (b \cdot x) = b \cdot x$, so that $gb \cdot x = b \cdot x$. Applying b^{-1} to both sides we get

$$b^{-1} \cdot (gb \cdot x) = b^{-1} \cdot (b \cdot x) \quad \Longrightarrow \quad b^{-1}gb \cdot x = x.$$

Hence $h := b^{-1}gb \in S(x)$ and therefore $g = bhb^{-1} \in bS(x)b^{-1}$. This shows that $S(b \cdot x) \subseteq bS(x)b^{-1}$

Suppose now that $g \in bS(x)b^{-1}$ be arbitrary. Thus $g = bhb^{-1}$ for some $h \in S(x)$, that is for some h such that $h \cdot x = x$. Then

$$g \cdot (b \cdot x) = bhb^{-1} \cdot (b \cdot x) = bh \cdot ((b^{-1}b) \cdot x) = bh \cdot x = b \cdot (h \cdot x) = b \cdot x.$$

Thus $g \in S(b \cdot x)$ and hence $bS(x)b^{-1} \subseteq S(b \cdot x)$.

It follows that $S(b \cdot x) = bS(x)b^{-1}$, as required.

(c) If $S(x)$ and $S(y)$ are conjugate subgroups of G, show that $|Gx| = |Gy|$.

Solution

We first need to establish the following general lemma:

Lemma. Let G be a group, $H \le G$ be a subgroup and let $u \in G$. Then $[G : H] = [G : uHu^{-1}]$.

Proof of Lemma. By definition $[G : H] = |G/H| = |\{gH : g \in G\}|$ and $[G : uHu^{-1}] = |G/uHu^{-1}| = |\{guHu^{-1} : g \in G\}|$. Thus it siffies to construct a bijection between the sets G/H and G/uHu^{-1}. Define $f : G/H \to uHu^{-1}$ by $f(gH) := ugu^{-1}uHu^{-1}$ for $g \in G$. Note first that f is well-defined. Indeed, if $gH = g'H$ then $g' = gh$ for some $h \in H$ and

$$ug'u^{-1}uHu^{-1} = ughu^{-1}uHu^{-1} = ugu^{-1}uhu^{-1}uHu^{-1} = ug'u^{-1}uHu^{-1}.$$

Thus f is well-defined. We claim that f is a bijection. For any $c \in G$ we have $cuHu^{-1} = u(u^{-1}cu)u^{-1}uHu^{-1} = f(u^{-1}cuH)$, so that f is onto. Suppose now that $f(g_1H) = f(g_2H)$ so that $ug_1u^{-1}uHu^{-1} = ug_2u^{-1}uHu^{-1}$. Hence $ug_2u^{-1} = ug_1u^{-1}z$ for some $z \in uHu^{-1}$, that is for some $h \in H$ we have

$$ug_2u^{-1} = ug_1u^{-1}uhu^{-1} = ug_1hu^{-1}.$$

Therefore $g_2 = g_1h$ and hence $g_1H = g_2H$. Thus f is one-to-one. We have verified that f is bijective so that $[G : H] = [G : uHu^{-1}]$ as claim. This completes the proof of the lemma. □

Now let x and y be as in part (c) of the problem. By the orbit-stabilizer formula (Lemma 3 in Ch 8.3) we have $|Gx| = [G : S(x)]$ and $|Gy| = [G : S(y)]$. Since $S(x)$ and $S(y)$ are conjugate in G, the Lemma implies that $[G : S(x)] = [G : S(y)]$ and hence $|Gx| = Gy|$, as required.

Ch. 8.4, Problem 2

Find all Sylow 2-subgroups of D_n, where n is odd, and show explicitly that they are conjugate.

Solution.

Let $n \ge 3$ be odd. Then $|D_n| = 2n$, so every Sylow 2-subgroup of D_n has order 2 and has the form $\langle g \rangle = \{1, g\}$ where $g \in D_n$ is an element of order 2. Thus to find all the Sylow 2-subgroups of D_n we need to find all elements of order 2 in D_n.

Recall that

$$D_n = \{1, a, a^2, \dots, a^{n-1}, b, ba, \dots ba^{n-1}\}$$

where $|a| = n$, $|b| = 2$ and $aba = b$.

Since $|a| = |\langle a \rangle| = n$ is odd, for every $g \in \langle a \rangle$ we have $|g| \big| n$ and hence $|g| \ne 2$.

We claim that $|ba^i| = 2$ for every $i = 0, 1, \dots, n-1$. Indeed, $aba = b$ implies $ab = ba^{-1}$ and $a^ib = a^{-i}b$ for all i. Hence

$$(ba^i)^2 = ba^iba^i = bba^{-i}a^i = b^2 = 1.$$

Since $ba^i \ne 1$, it follows that $|ba^i| = 2$ for $i = 0, 1, \dots, n-1$. Thus D_n has n elements of order 2 and, coirrespondingly, n Sylow 2-subgrops, namely, the subgroups $\langle ba^i \rangle = \{1, ba^i\}$ for $i = 0, \dots, n-1$. To see that they are all conjugate, it suffices to show that ba^i is conjugate to b for every $i = 0, \dots, n-1$.

Note that for every j we have $a^{-j}ba^j = ba^{2j} = b(a^2)^j$. Since n is odd and $gcd(n,2) = 1$, it follows that $\langle a\rangle = \langle a^2\rangle$. Thus for every $i = 0,\dots,n-1$ there exists j such that $a^i = a^{2j}$ and hence $ba^i = a^{-j}ba^j$ and $\langle ba^i\rangle = a^{-j}\langle b\rangle a^j$. Thus indeed all Sylow 2-subgroups of D_n are conjugate in D_n.

Ch. 8.4, Problem 3

If P is a Sylow p-subgroup of G, prove that P is the only Sylow p-subgroup of $N(P)$.

Solution.

Let $|G| = p^n m$ where $n \geq 1$ and $gcd(p,m) = 1$. Since $P \leq G$ is a Sylow P-subgroup of G, we have $|P| = p^n$. We have $P \leq N(P) \leq G$. Hence $|P|\big||N(P)|$ and $|N(P)|\big||G|$. Thus $p^n\big||N(P)|$ and $|N(P)|\big|p^n m$. Hence $|N(P)| = p^n m'$ where $m'|m$ and $gcd(p,m') = 1$.

Since $|P| = p^n$ and $P \leq N(P)$, it follows that P is a Sylow p-subgroup of $N(P)$. By definition, every subgroup is normal in its normalizer, and hence $P \lhd N(P)$. By the Second Sylow Subgroup Theorem every Sylow p-subgroup P' of $N(P)$ is conjugate to P in $N(P)$. Since $P \lhd N(P)$, this implies that $P' = P$. Hence P is the unique Sylow p-subgroup of $N(P)$, as claimed.

Ch. 8.4, Problem 4

Prove that every group of order 15 is cyclic.

Solution.

Let G be a group such that $|G| = 15 = 3\cdot 5$. Let n_3 be the number of Sylow 3-subgroups of G. Then by the Third Sylow Subgroup Theorem $n_3|5$ and $n_3 \equiv 1 \mod 3$. The condition $n_3|5$ implies that $n_3 = 1$ or $n_3 = 5$. The case $n_3 = 5$ is impossible since $5 \not\equiv 1 \mod 3$. Thus $n_3 = 1$. Let $H \leq G$ is the Sylow 3-subgroup of G, so that $|H| = 3$. Since for every $g \in G$ we have $|gHg^{-1}| = |H| = 3$ and $gHg^{-1} \leq G$ is also a Sylow 3-subgroup of G, the condition $n_3 = 1$ implies that $gHg^{-1} = H$. Hence $H \lhd G$ is normal in G.

Let n_5 be the number of Sylow 5-subgroups of G. Then by the Third Sylow Subgroup Theorem $n_5|3$ and $n_5 \equiv 1 \mod 5$. The condition $n_5|3$ implies that $n_5 = 1$ or $n_5 = 3$. The case $n_5 = 3$ is impossible since $3 \not\equiv 1 \mod 5$. Thus $n_5 = 1$. As above, this implies that if $K \leq G$ is a Sylow 5-subgroup (that is $|K| = 5$) then $K \lhd G$.

We claim that $H \cap K = \{1\}$. Indeed, suppose $a \in H \cap K$. Then, since $a \in H$, we have $|a|\big||H|$, that is $|a|\big|3$. Similarly, since $a \in K$, we have $|a|\big||K|$, that is $|a|\big|5$. Thence $|a| = 1$ and therefore $a = 1$. Thus indeed $H \cap K = \{1\}$.

Finally we have $|H|\cdot|K| = 3\cdot 5 = |G|$.

Thus $H \lhd G, K \lhd G$, $H \cap K = \{1\}$ and $|H|\cdot|K| = |G| < \infty$. Hence by Theorem 6 in Ch 2.8 we have $G \cong H \times K$. Since $|H| = 3$ is a prime, it follows that H is cyclic of order 3 and thus $H \cong \mathbb{Z}_3$. Similarly, since $|K| = 5$ is a prime, it follows that K is cyclic of order 5 and thus $K \cong \mathbb{Z}_5$. Thus $G \cong H \times K \cong \mathbb{Z}_3 \times \mathbb{Z}_5$. Since $gcd(3,5) = 1$, we have $\mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$. Therefore $G \cong \mathbb{Z}_{15}$, so that G is cyclic, as required.

Ch. 8.4, Problem 13

If $|G| = p^n m$ where $n \geq 1$, p is a prime and $p > m$, show that the Sylow p-subgroup of G is normal in G.

Solution.

Let n_p be the number of Sylow p-subgroups of G. By the 3-d Sylow Subgroup Theorem we know that $n_p|m$ and that $n_p \equiv 1 \mod p$.

Since $m < p$ and $n_p|m$, it follows that $1 \leq n_p \leq m < p$. Since we also know that $n_p \equiv 1 \mod p$, it follows that $n_p = 1$. Let P be a Sylow p-subgroup of G. Since for every $g \in G$ $g^{-1}Pg$ is also a Sylow p-subgroup of G and since $n_p = 1$, it follows that for every $g \in G$ $g^{-1}Pg = P$. Hence P is normal in G, as claimed.

Ch. 8.4, Problem 14

If $|G| = p^2q$ where p and q are primes, show that G is not simple.

Solution.

Suppose first that $p = q$. Then $|G| = p^3$ and G is a finite p-group. As was proved in class, every finite p-group has a nontrivial center, $Z(G) \neq \{1\}$. If $Z(G) \neq G$ then $Z(G) \lhd G$ and $Z(G) \neq G, Z(G) \neq \{1\}$, so that G is not simple. If $Z(G) = p^3$ then G is abelian. By the First Sylow Subgroup Theorem G has a subgroup H of order p. Then $H \neq \{1\}, H \neq G$ and $H \lhd G$ and hence G is not simple.

Suppose now that $p \neq q$. By the Third Sylow Subgroup Theorem $n_p|q$ and $n_p \equiv 1 \mod p$. Hence $n_p \in \{1, q\}$ and $n_p = 1 + pk$ for some integer k. If $n_p = 1$ then the Sylow p-subgroup of G is a proper normal subgroup in G and hence G is not simple.

Suppose now that $n_p = q$. Since $n_p - 1 = pk$, we have $q - 1 = pk$ and hence $p \leq q - 1$.

Again applying the Third Sylow Subgroup Theorem we get $n_q|p^2$ and $n_q \equiv 1 \mod q$. Thus $n_q \in \{1, p, p^2\}$. If $n_q = 1$, then he Sylow q-subgroup of G is a proper normal subgroup in G and hence G is not simple, as required.

If $n_q = p$ then the condition $n_q \equiv 1 \mod q$ implies $q|p - 1$ and hence $q \leq p - 1$. Since we already know that $p \leq q - 1$, this yields a contradiction.

Thus $n_q = p^2$. Hence $n_q \equiv 1 \mod q$ implies $q|(p^2 - 1)$, that is $q|(p - 1)(p + 1)$. Since q is a prime, it follows that $q|p - 1$ or $q|p + 1$.

If $q|p - 1$ then $q \leq p - 1$. Since we already know that $p \leq q - 1$, this again yields a contradiction.

Thus $q|p + 1$ and hence $q \leq p + 1$. Since we laready know that $p \leq q - 1$, we have $p + 1 \leq (q - 1) + 1 = q$. Thus $q \leq p + 1 \leq q$ and hence $q = p + 1$.

Since both p and q are primes and $q = p + 1$, the numbers p and q cannot both be odd. The only even prime is 2 and hence $p = 2$, $q = 3$. Therefore $|G| = p^2q = 2^2 \cdot 3 = 12$.

By Theorem 5 in Ch 8.4 every group of order 12 is isomorphic to one of $C_{12}, C_6 \times C_2, A_4, D_6$ or Q_6. None of these groups are simple and hence G is not simple, as required.

We can check that none of the groups in the above list are simple directly. Indeed, if $C_{12} = \langle x \rangle$ is cyclic of order 12, then $\langle x^2 \rangle$ has order 6 and is a proper normal subgroup of C_{12}.

Similarly, the subgroup $C_6 \times \{1\}$ is a subgroup of order 6 in the abelian group $C_6 \times C_2$ and thus is a proper normal subgroup.

We have seen in class that $V = \{\epsilon, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \leq A_4$ is a proper normal subgroup in A_4 (for example, because it has index 2).

Also, for $D_6 = \{1, a, \dots a^5, b, ba, \dots ba^5\}$ with $|a| = 6$, $|b| = 2$ and $aba = b$ the subgroup $\langle a \rangle$ has index 2 in D_6 and is therefore a proper normal subgroup.

Finally, for the group $Q_6 = \{1, a, \dots a^5, b, ba, \dots ba^5\}$, where $|a| = 6$, $aba = b$ and $b^2 = a^3$, the subgroup $\langle a \rangle$ has index 2 in D_6 and is therefore a proper normal subgroup.