Quiz 10 (Solutions); Friday, May 1, 2009

1. For each of the following statements indicate if it is true or false. You do not need to explain your answers here.

(1) If F is a field and $f(x) \in F[x]$ is a polynomial of degree $n \geq 1$ such that $f(x)$ has no roots in F then $f(x)$ is irreducible over F.
(2) If F is a field and $f(x) \in F[x]$ is a polynomial of degree 1 then $f(x)$ is irreducible over F.
(3) If R is a commutatrive ring and $f, g \in R[x]$ are such that $f \neq 0$, $g \neq 0$ then $deg(fg) = deg(f) + deg(g)$.
(4) If R is a commutative ring and $f(x) \in R[x]$ is a polynomial with $deg(f) = n \geq 1$ then $f(x)$ has at most n roots in R.
(5) The polynomial $f(x) = x^3 - 1$ is reducible over every field F.

Answers:

(1) False. This statement is only true in general for polynomials of degree $n = 2$ and $n = 3$ (and it was only proved in that case). A polynomial of degree ≥ 4 may have no roots but nonetheless be reducible. For example, consider $f(x) = (x^2+1)^2 = (x^2+1)(x^2+1) \in \mathbb{R}[x]$. This polynomial has no roots in $\mathbb{R}$ but is obviously reducible over $\mathbb{R}$.

(2) True.

(3) False.

This statement is only true if R is an integral domain. For example, take $f(x) = g(x) = \overline{2}x \in \mathbb{Z}_4[x]$. Then $deg(f) = deg(g) = 1$ but $fg = 0$ and so $deg(fg) \neq deg(f) + deg(g)$.

(4) False. This statement is only true if R is an integral domain.

The following example was a part of the last h/wk: the polynomial $f(x) = x^2 - x$ annihilates the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus has 4 roots in this ring. However $4 > 2 = deg(f)$.

(5) True. Indeed, $1 \in F$ is always a root of $f(x) = x^3 - 1$ and therefore $f(x) = x^3 - 1$ is reducible over every field F.