

Quiz 6 (Solutions); Friday, March 13, 2009

1. For each of the following statements indicate if it is true or false. You do not need to explain your answers here.

- (1) If G and H are countably infinite abelian groups where every nontrivial element has infinite order, then $G \cong H$.
- (2) The map $f : \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$, given by $f([0]_2) = [0]_4$ and $f([1]_2) = [2]_4$, is a homomorphism.
- (3) If G is a cyclic group of finite order $n \geq 2$ then $G \cong \mathbb{Z}_n$.
- (4) The groups $(\mathbb{R}, +)$ and (\mathbb{R}^+, \cdot) are isomorphic. Here $\mathbb{R}^+ = (0, \infty) = \{x \in \mathbb{R} : x > 0\}$.
- (5) The groups S_3 and \mathbb{Z}_6 are not isomorphic.

Answers:

(1) FALSE. For example, both \mathbb{Z} and $\mathbb{Z} \times \mathbb{Z}$ are countably infinite abelian groups where every nontrivial element has infinite order. However \mathbb{Z} is cyclic while $\mathbb{Z} \times \mathbb{Z}$ is not cyclic, so they are not isomorphic. No matter how many abstract properties two groups share, this does not, in general, imply that they are isomorphic. Abstract properties (being abelian, cyclic, finite, finitely generated, etc) can only be used to distinguish two groups, that is to show that they are NOT isomorphic. To prove that two groups are isomorphic one almost always needs to construct an explicit isomorphism.

(2) TRUE.

Since $f([0]_2) = [0]_4$, we obviously have $f([0]_2 + [0]_2) = f([0]_2) + f([0]_2)$ and $f([0]_2 + [1]_2) = f([0]_2) + f([1]_2)$, $f([1]_2 + [0]_2) = f([1]_2) + f([0]_2)$.

Note that \mathbb{Z}_2 is a cyclic group of order 2 and that f maps $[1]_2$ (and element of order 2 in \mathbb{Z}_2) to $[2]_4$, an element of order 2 in \mathbb{Z}_4 . Therefore we also have $f([1]_2 + [1]_2) = [0]_4 = f([1]_2) + f([1]_2)$. Thus $f(a + b) = f(a) + f(b)$ for any $a, b \in \mathbb{Z}_2$ and f is a homomorphism.

(3) TRUE. (See Example 13 in Ch. 2.5)

(4) TRUE. These groups are isomorphic, as was proved in class. Specifically, the exponential map $\exp : \mathbb{R} \rightarrow \mathbb{R}^+, x \mapsto e^x$ is an isomorphism between them.

(5) TRUE. Indeed, the groups S_3 and \mathbb{Z}_6 are not isomorphic because \mathbb{Z}_6 is abelian while S_3 is not abelian.