

## Quiz 7 (Solutions); Friday, March 20, 2009

1. For each of the following statements indicate if it is true or false. You do not need to explain your answers here.

- (1) If  $G$  is a group such that  $|G| = p \geq 2$  is a prime then  $G$  is abelian.
- (2) If  $H \leq G$  and  $gH = Hg$  for every  $g \in G$  then  $G$  is abelian.
- (3) If  $H = \langle (1\ 2\ 3) \rangle \leq S_4$  then  $[S_4 : H] = 3$ .
- (4) If  $g \in G$ ,  $|G| = 100$  and  $[G : \langle g \rangle] = 5$  then  $|g| = 20$ .
- (5) If  $|G| = 50$ , then for every  $g \in G$  such that  $g \neq 1$  we have  $g^3 \neq 1$ .

### Answers:

(1) True. Indeed, if  $|G|$  is a prime then  $G$  is cyclic (by Corollary 3 in Ch. 2.6) and therefore abelian.

(2) False. For example, if  $G$  is a nonabelian group (e.g.  $G = S_3$ ) and  $H = G$  then  $gH = Hg = G$  for every  $g \in G$ . Also, one can take  $H = \{1\}$  in this case and then  $gH = Hg = \{g\}$ , even though  $G$  is non-abelian.

(3) False. Indeed, we have  $|H| = |\langle (1\ 2\ 3) \rangle| = |(1\ 2\ 3)| = 3$  and hence by Lagrange's Theorem  $[G : H] = \frac{|G|}{|H|} = \frac{24}{3} = 8 \neq 3$ .

(4) True. Indeed, by Lagrange's Theorem

$$|g| = |\langle g \rangle| = \frac{|G|}{[G : \langle g \rangle]} = \frac{100}{5} = 20.$$

(5) True. Indeed, suppose there is  $g \neq 1$  in  $G$  such that  $g^3 = 1$ . Since 3 is a prime, it follows that  $|g| = 3$ . However, by Corollary 1 in Ch. 2.6 we have  $|g| \mid |G|$ , that is  $3 \mid 50$ , which is a contradiction.