Quiz 9 (Solutions); Friday, April 10, 2009

1. For each of the following statements indicate if it is true or false. You do not need to explain your answers here.

(1) The subset $\{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{Z}\} \subseteq \mathbb{Z}[x]$ is a subring of $\mathbb{Z}[x]$.

(2) If R is a ring with characteristic $m \geq 1$ then for every $r \in R$ we have $r^m = 1$.

(3) The subset $GL(2, \mathbb{R}) \subseteq M_2(\mathbb{R})$ is a subring of $M_2(\mathbb{R})$.

(4) If R is a ring and R^* is the set of all units in R then $(R^*, \cdot\ )$ is a group.

(5) If R is a ring and $a, b \in R$ then $(a + b)^2 = a^2 + 2ab + b^2$ in R.

Answers:

(1) False. Denote $S := \{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{Z}\} \subseteq \mathbb{Z}[x]$. The subset S of $\mathbb{Z}$ is not closed under multiplication. For example, $x^2, x \in S$ but $x^2 \cdot x = x^3 \notin S$.

(2) False. By definition, if $char(R) = m \geq 1$ then for every $r \in R$ we have $mr = 0$. However, the definition of characteristic says nothing about multiplicative powers of elements of R. For a concrete counter-example, consider $R = \mathbb{Z}_5$. We have $char(\mathbb{Z}_5) = 5$, but $\overline{2}^5 = \overline{32} = \overline{2} \neq \overline{1}$ in $\mathbb{Z}_5$.

(3) False. The subset $GL(2, \mathbb{R}) \subseteq M_2(\mathbb{R})$ is not a subgroup of the additive group of $M_2(\mathbb{R})$ because, for example, the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin GL(2, \mathbb{R})$.

(4) True.

(5) False. As was explained in class, we always have $(a + b)^2 = a^2 + ab + ba + b^2$. However, if the ring R is non-commutative and $a, b \in R$ are such that $ab \neq ba$ then $ab + ba \neq 2ab$ and $(a + b)^2 = a^2 + ab + ba + b^2 \neq a^2 + 2ab + b^2$.