

Math 432 Exam 1 (Solutions), Friday, October 3, 2008

1.

For each of the following items indicate if the statement is true or false. You do not need to justify your answers here.

- (1) For every set A we have $o(A) < o(P(A))$.
- (2) If L is a chain then L is a distributive lattice.
- (3) If $A \subseteq B$ such that the set $B - A$ is infinite then $o(A) < o(B)$.
- (4) If A is countable then the set $P(A)$ is uncountable.
- (5) If d, e are cardinal numbers such that $d \leq e$ then $de = e$.

Answers:

- (1) True.
- (2) True.
- (3) False.
- (4) False. (The statement is true if A is countably infinite but false if A is finite).
- (5) False (e.g. when d and e are nonzero finite cardinals).

2.

- (a) Give an example of a partially ordered set where every chain has an upper bound and where there exist two distinct maximal elements.
- (b) Give an example of a countably infinite chain which has both a top element and a bottom element.

Solution.

- (a) Take the set $A = \{a, b\}$ that is totally unordered, that is where the partial order \leq on A consists only of $a \leq a$ and $b \leq b$.
- (b) Take, for example $A = \mathbb{A} \cap [0, 1]$.

3. Let

$$A = \{m_0 + m_1\pi + \cdots + m_k\pi^k \mid k \geq 0, m_i \in \mathbb{Q}\} \subseteq \mathbb{R}.$$

Prove that the set A is countable.

Solution.

For each integer $k \geq 1$ put

$$A_k = \{m_0 + m_1\pi + \cdots + m_k\pi^k \mid m_i \in \mathbb{Q}\} \subseteq \mathbb{R}.$$

Define $f_k : \mathbb{Q}^{k+1} \rightarrow A_k$ by $f_k(m_0, m_1, \dots, m_k) = m_0 + m_1\pi + \cdots + m_k\pi^k$ for any $(m_0, m_1, \dots, m_k) \in \mathbb{Q}^{k+1}$. The set \mathbb{Q} is countable and hence \mathbb{Q}^{k+1} is countable. The function f_k is obviously "onto" and therefore A_k is countable also.

We have $A = \bigcup_{k=1}^{\infty} A_k$, a union of a countable family of countable sets, and hence A is countable as well.

4. Let d_1 and e be infinite cardinals and let $d = 2^{d_1}$. Suppose that $d \geq 2^e$. Prove that $d^e = d^2$.

Solution.

This problem is almost identical to the h/wk problem Ch. 2.6 no. 6 that was done in class.

Since $2^e \leq d$, we have

$$\begin{aligned} d^e &= (2^{d_1})^e = 2^{d_1 e} = (2^e)^{d_1} \leq d^{d_1} = \\ &= (2^{d_1})^{d_1} = 2^{d_1 d_1} = 2^{d_1} = d = dd = d^2. \end{aligned}$$

Thus $d^e \leq d^2$. On the other hand, $2 \leq e$ implies $d^2 \leq d^e$. Hence $d^2 = d^e$.

Note. As I explained in class, although we do know that $d = 2^{d_1} \leq 2^e$, we cannot assume that this implies that $d_1 \leq e$.

5.

(a) State Zorn's Lemma.

(b) Let L be a nonempty lattice where every chain has an upper bound. Prove that L contains a top element.

Solution.

(a) If L is a nonempty partially ordered set where every chain has an upper bound in L , then L contains a maximal element.

(b) By Zorn's Lemma L contains a maximal element m . We claim that m is a top element in L .

Let $a \in L$ be arbitrary. Since L is a lattice, there exists $y = a \vee m \in L$.

Thus $a \leq y, m \leq y$. Since $m \in L$ is maximal, $m \leq y$ implies $m = y$. Thus $a \leq y = m, a \leq m$. Since $a \in L$ was arbitrary, this shows that $m \in L$ is a top element.