Math 432 Exam 2 (solutions), Monday, November 17, 2008

1.

For each of the following items indicate if the statement is true or false. You do not need to justify your answers here.

(1) Every finite nonempty partially ordered set is well ordered.
(2) There does not exist a separable metric space X with $o(X) > c$.
(3) Whenever $f : X \to Y$ is uniformly continuous and U is an open subset of X then $f(U)$ is an open subset of Y.
(4) Every nonempty open set can be represented as a union of open balls.
(5) Every subsequence of a Cauchy sequence is itself a Cauchy sequence.

Answers:

(1) False. E.g. any finite partially ordered set that is not a chain is not well ordered.
(2) True.
(3) False. E.g. the function $f : [0, 1] \to \mathbb{R}$, $f(x) = x$, is uniformly continuous. The set $U = [0, 1]$ is an open subset of $X = [0, 1]$ but $f(U) = [0, 1]$ is not an open subset of $\mathbb{R}$.
(4) True.
(5) True.

2.

(a) Let L be a chain where every countable subset is well-ordered. Prove that L is well-ordered.

(b) Give an example of a countably infinite well-ordered set that is not order-isomorphic to $(\mathbb{N}, \leq)$. Explain why your example has the required property.

Solution.

(a) Suppose, on the contrary, that L is not well-ordered. Then by Theorem 17 in Ch. 3.1 L contains an infinite descending sequence

$$a_0 > a_1 > a_2 > \dots$$

Put $A = \{a_n | n \geq 1\}$. Then A is a nonempty countable subset of L that does not contain a minimal element. This contradicts our assuption that every countable subset of L is well-ordered.

(b) Consider the set $X = \mathbb{N} \cup \{*\}$. Define a partial order on X by extending the standard ordering on $\mathbb{N}$ by setting $n \leq *$ for every $n \in \mathbb{N}$ and by setting $* \leq *$. Then $(X, \leq)$ is a countably infinite well-ordered set. Moreover, the segment $S(*)$ of X is equal to $\mathbb{N}$ with the standard order on $\mathbb{N}$. Therefore $(X, \leq)$ and $(\mathbb{N}, \leq)$ are not order-isomorphic since we know that a well-ordered set cannot be isomorphic to one of its segments (Theorem 20 in Ch. 3.3).

3.

Let X be a metric space and let $Y \subseteq X$ be a subset such that Y is separable (considered as a metric space with the metric restricted from X). Prove that the closure $\overline{Y}$ of Y in X is separable.

Solution. (Note that this was one of the homework problems).

Let A be a countable dense subset of Y, so that the closure of A in Y is equal to Y. We claim that the closure of A in X is equal to $\overline{Y}$.

Indeed, let $z \in \overline{Y}$ be arbitrary. Then there exists a sequence $y_n \in Y$ such that $\lim_{n\to\infty} y_n = z$. Since A is dense in Y, for every $n \geq 1$ there exists $a_n \in A$ such that $d(y_n, a_n) \leq 1/n$. It is easy to see then that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} y_n = z,$$

so that z belongs to the closure $\overline{A}$ of A in X.

Indeed, for every $\epsilon > 0$ we can find $N \geq 1$ such that for all $n \geq N$ we have $1/n < \epsilon/2$ and $d(y_n, z) < \epsilon/2$. Then for every $n \geq N$

$$d(a_n, z) \leq d(a_n, y_n) + d(y_n, z) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so that $\lim_{n\to\infty} a_n = z$, as required.

This shows that $\overline{Y} \subseteq \overline{A}$. Since, $A \subseteq Y$, the inclusion $\overline{A} \subseteq \overline{Y}$ is obvious. Hence $\overline{A} = \overline{Y}$. Thus $\overline{Y}$ has a dense countable subset (namely A) and $\overline{Y}$ is separable as required.

4.

Let X be a metric space where every countable closed subset is complete. Prove that X is complete.

Solution. (Note that this was one of the homework problems that was also done in class on Friday, November 14).

First we prove:

Lemma. Let a_n be a Cauchy sequence in a metric space Y and let $A = \{a_n | n \geq 1\}$. Then:

(1) If the sequence a_n does not have a limit in Y, then $\overline{A} = A$.

(2) If the sequence a_n converges to a limit y in Y, that is $y = \lim_{n\to\infty} a_n$, then $\overline{A} = A \cup \{y\}$.

Proof of the Lemma.

We know that $A \subseteq \overline{A}$. Suppose $A \neq \overline{A}$, that is there exists $y \in \overline{A} - A$. Then y is the limit of a sequence of elements from A, which implies that there exists a subsequence a_{n_i} of a_n such that $\lim_{i\to\infty} a_{n_i} = y$. Since a_n is a Cauchy sequence, it follows that $y = \lim_{n\to\infty} a_n$.

It follows that $\overline{A} \subseteq A \cup \{y\}$, where $y = \lim_{n\to\infty} a_n$ when a_n converges, and that $\overline{A} = A$ when a_n does not converge. It is obvious that if a_n

converges and $y = \lim_{n\to\infty} a_n$ then $y \in \overline{A}$. Thus $\overline{A} = A \cup \{y\}$, where $y = \lim_{n\to\infty} a_n$ when a_n converges, and that $\overline{A} = A$ when a_n does not converge. This proves the Lemma.

Now let X be as in the problem and let x_n be a Cauchy sequence in X. Let $A = \{x_n | n \geq 1\}$. The Lemma implies that either $\overline{A} = A$ or $\overline{A} = A \cup \lim_{n\to\infty} x_n$. In either case $\overline{A}$ is countable. Since $\overline{A}$ is a closed and countable subset of X, by assumption $\overline{A}$ is complete. Since x_n is a Cauchy sequence in $\overline{A}$, there exists $y \in \overline{A}$ such that $\lim_{n\to\infty} a_n = y$ in $\overline{A}$ and hence $\lim_{n\to\infty} a_n = y$ in X. Thus X is complete as required.

5.

(a) Give an example of an uncountable metric space X such that for every metric space Y and every function $f : X \to Y$ the function f is continuous. Explain why your example has the required property.

(b) Give an example of a countable family $(U_n)_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$ such that their intersection $\cap_{n=1}^{\infty} U_n$ is not open in $\mathbb{R}$.

Solution.

(a) Let X be any uncountable set (e.g. $X = \mathbb{R}$ as a set) with the discrete metric $d(x, y) = 1$ when $x \neq y$, $x, y \in X$ and $d(x, x) = 0$ for $x \in X$. Then points are open in X and therefore every subset of X is open.

Thus if Y is another metric space and $f : X \to Y$ is a function, then for any subset V of Y (in particular, for any open subset V of Y) the set $f^{-1}(V) \subseteq X$ is open in X. Hence f is continuous.

(b) Take $U_n = (-1/n, 1/n)$ for $n \geq 1$. Then each U_n is open in $\mathbb{R}$ but

$$\cap_{n=1}^{\infty} U_n = \cap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$$

is not open in $\mathbb{R}$.