Homework 12, due Wednesday, December 10, 2008

1. A topological space X is said to be *Hausdorff* if for any $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Prove that if (M, d) is a metric space, then, considered as a topological space, M is Hausdorff.

2. For each subset $A \subseteq \mathbb{R}$ put

$$U_A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \in A\} \subseteq \mathbb{R}^2.$$

Let $\mathcal{T}$ be the set of all subsets U of X of the form

$$\mathcal{T} = \{U_A : A \subseteq \mathbb{R} \text{ is open in } \mathbb{R}\}.$$

(a) Prove that $\mathcal{T}$ is a topology on the set $\mathbb{R}^2$.

(b) Prove that there does not exist a metric d on $\mathbb{R}$ such that the topology $\mathcal{T}$ is the metric space topology for $(\mathbb{R}^2, d)$.

(c) Let $a_n = (1, 0)$ when $n \geq 1$ is even and $a_n = (-1, 0)$ when $n \geq 1$ is odd. Is it true that $\lim_{n\to\infty} a_n = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ in $(\mathbb{R}^2, \mathcal{T})$? Explain why or why not.

(d) Let X_1 be $\mathbb{R}^2$ with the topology $\mathcal{T}$ defined above and let X_2 be $\mathbb{R}^2$ with the topology coming from the standard Euclidean metric on $\mathbb{R}^2$.

Is the identity map $Id : X_1 \to X_2$ continuous? Is the identity map $Id : X_2 \to X_1$ continuous?

(e) Let X_1 be as in part (d) and let $f : X_1 \to \mathbb{R}$ be a continuous function, where $\mathbb{R}$ is equipped with the topology coming from the standard metric on $\mathbb{R}$.

Prove that $f(1, 0) = f(-1, 0)$.