

7. Tangent Vectors on Manifolds

1. Tangent Vectors to M [Frankel, p.23-30]

a) The tangent vector to a curve $\gamma(t)$ is written $\frac{d\gamma}{dt}$ or $\frac{d}{dt}$ or $\dot{\gamma}(t)$. It acts as a first order differential operator on functions $f : M \rightarrow \mathbb{R}$, by the rule:

$$\frac{d}{dt}(f) = \frac{d}{dt}(f(\gamma(t))).$$

This is how we define tangent vectors on manifolds: as differential operators.

b) For a chart $(U, \phi_U = (x^1, \dots, x^n))$,

$$\frac{d}{dt} = \sum_i \frac{dx^i}{dt} \frac{\partial}{\partial x^i}$$

by the Chain Rule, where the tangent vector $\frac{\partial}{\partial x^i}$ is tangent to the i th coordinate curve.

The $\frac{\partial}{\partial x^i}|_p, 1 \leq i \leq n$, are the *coordinate basis vectors* for the *tangent space* M_p . M_p is a vector space, consisting of all tangent vectors to curves through p .

2. Example:

$M = \mathbb{R}^2 = \{(x^1, x^2)\}$. Set $y^1 = x^1 x^2, y^2 = (x^1)^2 - (x^2)^2$. When $(x^1, x^2) \neq (0, 0)$, we have

$$\begin{aligned} \det \frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} &= \det \begin{pmatrix} x^2 & x^1 \\ 2x^1 & -2x^2 \end{pmatrix} \\ &= -2((x^2)^2 + (x^1)^2) \neq 0 \end{aligned}$$

so that $\phi = (y^1, y^2)$ gives a chart on some open set \tilde{U} around any (x^1, x^2) except $(0, 0)$.

a) How big can we take \tilde{U} ? Is ϕ one to one on $\{(x^1, x^2) \neq (0, 0)\}$? Is ϕ one to one on $\{(x^1, x^2); x^1 > 0\}$? Is ϕ one to one on some open set around $(0, 0)$?

b) By the Chain Rule,

$$\frac{\partial}{\partial y^i} = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

$$\left(\frac{\partial}{\partial y^1} \quad \frac{\partial}{\partial y^2} \right) = \left(\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \right) \left(\frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} \right)$$

and so

$$\begin{aligned} \left(\frac{\partial}{\partial y^1} \quad \frac{\partial}{\partial y^2} \right) &= \left(\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \right) \left(\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} \right)^{-1} \\ &= \left(\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \right) \begin{pmatrix} x^2 & x^1 \\ 2x^1 & -2x^2 \end{pmatrix}^{-1} \\ &= \frac{1}{-2((x^1)^2 + (x^2)^2)} \left(\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \right) \begin{pmatrix} -2x^2 & -x^1 \\ -2x^1 & x^2 \end{pmatrix} \end{aligned}$$

At p_0 with $(x^1, x^2)(p_0) = (1, 1)$:

$$\left(\frac{\partial}{\partial y^1} \quad \frac{\partial}{\partial y^2} \right) = -\frac{1}{4} \left(\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \right) \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

Therefore, at p_0 , $\frac{\partial}{\partial y^1} = \frac{1}{2} \frac{\partial}{\partial x^1} + \frac{1}{2} \frac{\partial}{\partial x^2}$ and

$$\frac{\partial}{\partial y^2} = \frac{1}{4} \frac{\partial}{\partial x^1} - \frac{1}{4} \frac{\partial}{\partial x^2}.$$

Draw $\frac{\partial}{\partial y^1}$ and $\frac{\partial}{\partial y^2}$ at p_0 :

c) As above, on the overlap of two coordinate patches, the Jacobian is the change of basis matrix between the coordinate bases.

3. Vector fields on M

A *vector field* Y is a choice of tangent vector $Y_p \in M_p$ for each $p \in M$. In coordinates,

$$Y = \sum_{i=1}^n Y^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}.$$

We require the functions $Y^i(x^1, \dots, x^n)$ to be smooth.