1) $M = U = \mathbb{R}^2 = \{(x^1, x^2)\}$. Set $y^1 = (x^1)^2 + (x^2)^2$, $y^2 = x^1 x^2$.

ⓐ For what point(s) (x^1, x^2) can you NOT use (y^1, y^2) as coordinates on some open set around (x^1, x^2)? Explain.

$$\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 2x^1 & 2x^2 \\ x^2 & x^1 \end{pmatrix}$$

So (y^1, y^2) will NOT work as coordinates around a point on these lines.

$$\det = 2\left[(x^1)^2 - (x^2)^2\right] = 0 \text{ on lines } x^2 = \pm x^1.$$ (see 2(c), Lecture 8)

②

ⓑ Let p_0 be the point with $(x^1, x^2)(p_0) = (1, \frac{1}{2})$. Express $\frac{\partial}{\partial y^1}\big|_{p_0}$ & $\frac{\partial}{\partial y^2}\big|_{p_0}$ as linear combinations of $\frac{\partial}{\partial x^1}\big|_{p_0}$ & $\frac{\partial}{\partial x^2}\big|_{p_0}$.

$$\left(\frac{\partial}{\partial y^1} \;\; \frac{\partial}{\partial y^2}\right) = \left(\frac{\partial}{\partial x^1} \;\; \frac{\partial}{\partial x^2}\right) \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} = \left(\frac{\partial}{\partial x^1} \;\; \frac{\partial}{\partial x^2}\right)\left(\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}\right)^{-1}$$

at $p_0 = (1, \frac{1}{2})$:

$$\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right) = \left(\frac{\partial}{\partial x^1} \;\; \frac{\partial}{\partial x^2}\right)\begin{pmatrix} 2 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}^{-1} = \left(\frac{\partial}{\partial x^1} \;\; \frac{\partial}{\partial x^2}\right)\begin{pmatrix} 2/3 & -2/3 \\ -1/3 & 4/3 \end{pmatrix}$$

$$\frac{\partial}{\partial y^1} = \frac{2}{3}\frac{\partial}{\partial x^1} - \frac{1}{3}\frac{\partial}{\partial x^2}$$

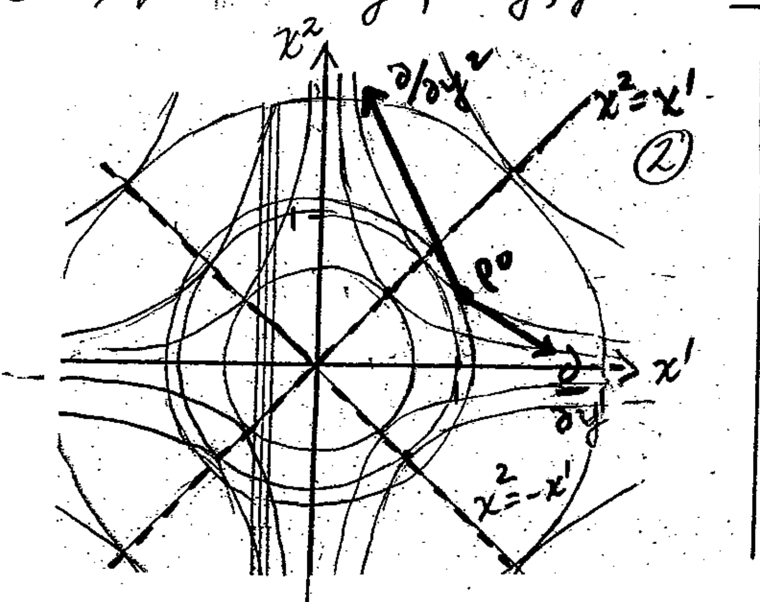
$$\frac{\partial}{\partial y^2} = -\frac{2}{3}\frac{\partial}{\partial x^1} + \frac{4}{3}\frac{\partial}{\partial x^2}$$

②

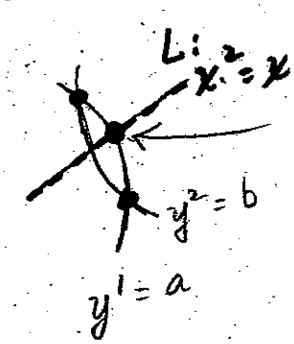
ⓒ Sketch the curves $y^1 = \frac{1}{2}, 1, 2$, & $y^2 = 0, \pm\frac{1}{2}, \pm 1, \pm 2$.

ⓓ Sketch the tangent vectors $\frac{\partial}{\partial y^1}\big|_{p_0}$, $\frac{\partial}{\partial y^2}\big|_{p_0}$ from ⓑ.

ⓔ Explain why $\phi = (y^1, y^2)$ is not 1-1 on any open set around a point from ⓐ.



ⓔ



Arbitrarily close to a point on L, you always have pairs of points with the same (y^1, y^2) — since reflection across L takes (x^1, x^2) to (x^2, x^1), so does not change (y^1, y^2).

②

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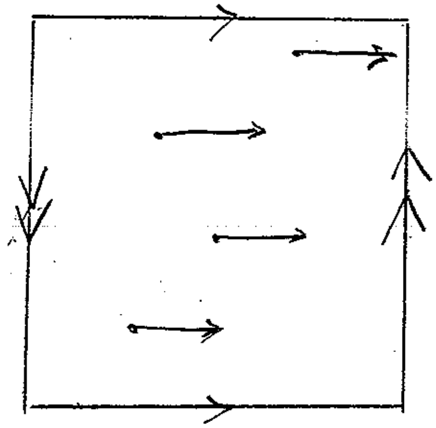
NAME: *Answers*

2/ Convince me by a sketch that the Klein bottle has a never-zero vector field:

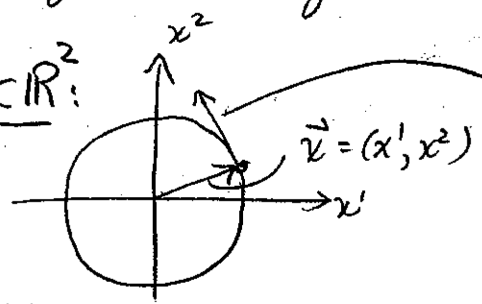


(3)

3/ Find a never-zero vector field on any odd-dimensional sphere S^{2n-1}. (Hint ①: $S^{2n-1} = \{\vec{x} = (x^1, x^2, \dots, x^{2n-1}, x^{2n}) \in \mathbb{R}^{2n} : \|\vec{x}\| = 1\}$. Then $(S^{2n-1})_{\vec{x}} = \{\vec{y} \in \mathbb{R}^{2n} : \vec{x}\cdot\vec{y} = 0\}$. For each $\vec{x} \in S^{2n-1}$, you must find such a $\vec{y} \neq \vec{0}$. Hint ②: Try it for $S^1 \subset \mathbb{R}^2$ first.)

(4)

<u>$S^1 \subset \mathbb{R}^2$:</u>



$\vec{y}_{\vec{x}} = (-x^2, x^1)$

<u>$S^{2n-1} \subset \mathbb{R}^{2n}$:</u> At each $\vec{x} = (x^1, x^2, \dots, x^{2n-1}, x^{2n})$,

take:

$$\vec{y}_{\vec{x}} = (-x^2, x^1, \dots, -x^{2n}, x^{2n-1}).$$

- $\vec{y}_{\vec{x}} \cdot \vec{x} = 0$, so $\vec{y}_{\vec{x}} \in (S^{2n-1})_{\vec{x}}$.
- $\|\vec{y}_{\vec{x}}\| = 1$, so $\vec{y}_{\vec{x}} \neq \vec{0}$.

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