THE NON-AMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

ILYA KAPOVICH

Abstract. We show that if H is a quasiconvex subgroup of infinite index in a non-elementary hyperbolic group G then the Schreier coset graph for G relative to H is non-amenable.

1. Introduction

A connected graph of bounded degree X is *non-amenable* if X has nonzero Cheeger constant, or, equivalently, if the spectral radius of the simple random walk on X is less than one (see Section 2 below for more precise definitions). Non-amenable graphs play an increasingly important role in the study of various probabilistic phenomena, such as random walks, harmonic analysis, Brownian motion and percolations, on graphs and manifolds (see for example [2, 5, 6, 7, 15, 17, 18, 24, 30, 43, 44, 62, 71, 72]) as well as in the study of expander families of finite graphs (see for example [52, 66, 67]).

It is well-known that a finitely generated group G is non-amenable if and only if some (any) Cayley graph of G with respect to a finite generating set is non-amenable. Word-hyperbolic groups are non-amenable unless they are virtually cyclic and thus their Cayley graphs provide a large and interesting class of non-amenable graphs. In this paper we investigate non-amenability of Schreier coset graphs corresponding to subgroups of hyperbolic groups.

We recall the definition of a Schreier coset graph:

Definition 1.1. Let G be a group and let $\pi : A \to G$ be a map where A is a finite alphabet such that $\pi(A)$ generates G (we refer to such A as a *marked finite generating set* or just *finite generating set* of G). Let $H \le G$ be a subgroup of G. The *Schreier coset graph* (or the *relative Cayley graph*) $\Gamma(G, H, A)$ for G relative to H with respect to A is an oriented labeled graph defined as follows:

1. The vertices of $\Gamma = \Gamma(G, H, A)$ are precisely the cosets of H in G, that is $V\Gamma := \{Hg \,|\, g \in G\}$.
2. The set of positively oriented edges of $\Gamma(G, H, A)$ is in one-to-one correspondence with the set $V\Gamma \times A$. For each pair $(Hg, a) \in V\Gamma \times A$

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there is a positively oriented edge in $\Gamma(G, H, A)$ from Hg to $Hg\pi(a)$ labeled by the letter a.

Thus the label of every path in $\Gamma(G, H, A)$ is a word in the alphabet $A \cup A^{-1}$. The graph $\Gamma(G, H, A)$ is connected since $\pi(A)$ generates G. Moreover, $\Gamma(G, H, A)$ comes equipped with the natural simplicial metric d obtained by giving every edge length one.

We can identify the Schreier graph with 1-skeleton of the presentation complex of G corresponding to any presentation of G of the form $G = \langle A \,|\, R \rangle$. It is also easy to see that if M is a closed Riemannian manifold and $H \leq G = \pi_1(M)$, then the Schreier graph $\Gamma(G, H, A)$ is quasi-isometric to the covering space of M corresponding to H.

If H is normal in G and $G_1 = G/H$ is the quotient group, then $\Gamma(G, H, A)$ is exactly the Cayley graph of the group G_1 with respect to A. In particular, if $H = 1$ then $\Gamma(G, 1, A)$ is the standard *Cayley graph of G with respect to A*, denoted $\Gamma(G, A)$.

The notion of a *word-hyperbolic group* was introduced by M.Gromov [40] and has played a central role in Geometric Group Theory in the last fifteen years. A subgroup H of a word-hyperbolic group G is said to be *quasiconvex* in G if for any finite generating set A of G there is $\epsilon \geq 0$ such that every geodesic in $\Gamma(G, A)$ with both endpoints in H is contained in the ϵ-neighborhood of H in $\Gamma(G, A)$. Quasiconvex subgroups are closely related to geometric finiteness in the Kleinian group context [69]. They enjoy a number of particularly good properties and play an important role in hyperbolic group theory and its applications (see for example [3, 4, 8, 31, 34, 35, 36, 37, 38, 42, 45, 46, 48, 51, 53, 55, 61, 70]).

Our main result is the following:

Theorem 1.2. *Let G be a non-elementary word-hyperbolic group with a marked finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. Then the Schreier coset graph $\Gamma(G, H, A)$ is non-amenable.*

The study of Schreier graphs arises naturally in various generalizations of J.Stallings' theory of ends of groups [23, 29, 60, 61, 63]. The case of virtually cyclic (and hence quasiconvex) subgroups of hyperbolic groups is particularly important to understand in the theory of JSJ-decomposition for hyperbolic groups originally developed by Z.Sela [65] and later by B.Bowditch [10] (see also [59, 23, 28, 64] for various generalizations of the JSJ-theory). A variation of the Følner criterion of non-amenability (see Proposition 2.3 below), when the Cheeger constant is defined by taking the infimum over all finite subsets containing no more than a half of all the vertices, is used to define an important notion of *expander families* of finite graphs. Most known sources of expander families involve taking Schreier coset graphs corresponding to subgroups of finite index in a group with Kazhdan property (T) (see [52, 66, 67] for a detailed exposition on expander families and their connections with non-amenability).

Since non-amenable graphs of bounded degree are well-known to be *transient* with respect to the simple random walk, Theorem 1.2 implies that $\Gamma(G, H, A)$ is also transient. This was first shown in [49] by more elementary means for the case when G is torsion-free hyperbolic $H \leq G$ is quasiconvex of infinite index.

It was originally stated by M.Gromov [40] and proved by R.Foord [27] and I.Kapovich [49] that for any quasiconvex subgroup H in a hyperbolic group G with a finite generating set A the coset graph $\Gamma(G, H, A)$ is a hyperbolic metric space. A great deal is known about random walks on hyperbolic graphs, but most of these results assume some kind of non-amenability. Thus Theorem 1.2 together with hyperbolicity of $\Gamma(G, H, A)$ and a result of A.Ancona [2] (see also the [72]) immediately imply:

Corollary 1.3. *Let G be a non-elementary word-hyperbolic group with a finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G and let Y be the Schreier coset graph $\Gamma(G, H, A)$.*

Then:

1. *The trajectory of almost every simple random walk on Y converges in the topology of $Y \cup \partial Y$ to some point in ∂Y (where ∂Y is the hyperbolic boundary).*
2. *There Martin boundary of a the simple random walk on X is homeomorphic to the hyperbolic boundary ∂X and the Martin compactification $\hat{X}$ for the simple random walk on X is homeomorphic to the hyperbolic compactification $X \cup \partial X$.*

The statement of Theorem 1.2 is easy to illustrate for the case of a free group. Suppose $F = F(a, b)$ is free and $H \leq F$ is a finitely generated subgroup of infinite index (which is therefore quasiconvex [68]). Put $A = \{a, b\}$. Then the Schreier graph $Y = \Gamma(F, H, A)$ looks like a finite graph with several infinite tree-brunches attached to it (the "brunches" are 4-regular trees except for the attaching vertices). In this situation it is easy to see that Y has positive Cheeger constant and so Y is non-amenable. Alex Lubotzky and Andrzej Zuk pointed out to the author that if G is a group with Kazhdan property (T) then for any subgroup H of infinite index in G the Schreier coset graph for G relative to H is non-amenable. There are many examples of word-hyperbolic groups with Kazhdan property (T) and in view of Theorem 1.2 it would be particularly interesting to investigate if they can possess non-quasiconvex finitely generated subgroups.

Non-amenability of graphs is closely related to co-growth. Thus we also obtain the following fact.

Corollary 1.4. *Let $G = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_m \rangle$ be a non-elementary hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n representing elements of H. Let b_n be the number of all words in A of length n representing elements of H.*

Then

$$\limsup_{n\to\infty} \sqrt[n]{a_n} < 2k-1$$

and

$$\limsup_{n\to\infty} \sqrt[n]{b_n} < 2k.$$

Theorem 1.2 is used in [50, 9] to obtain results about "generic-case" complexity of the membership problem as well as about measures of some natural subsets of free groups.

It is easy to see that the statement of Theorem 1.2 need not hold for finitely generated subgroups which are not quasiconvex. For example, a celebrated construction of E.Rips [58] states that for any finitely presented group Q there is a short exact sequence

$$1 \to K \to G \to Q \to 1,$$

where G is non-elementary torsion-free word-hyperbolic and where K can be generated by two elements (but K is not finitely presentable). If Q is chosen to be non-amenable, then the Schreier graph for G relative to H is non-amenable. Finitely presentable and even hyperbolic examples are also possible. For instance, if F is a free group of finite rank and $\phi : F \to F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \,|\, t^{-1}ft = \phi(f) \text{ for all } f \in F\rangle$$

is word-hyperbolic [8, 12]. In this case $G/F \simeq \mathbb{Z}$ and thus amenable.

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2. Non-amenability for graphs

Let X be a connected graph of bounded degree. We will denote by $\rho(X)$ the *spectral radius* of X which can be defined as

$$\rho(X) := \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$$

where x, y are two vertices of X and $p^{(n)}(x,y)$ is the probability that a simple random walk starting at x will end up at y in n steps. It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x, y.

Definition 2.1 (Amenability for graphs)**.** A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *non-amenable* if $\rho(X) < 1$.

It is also well-known that non-amenability of X implies that X is transient (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for the comprehensive background information about random walks on graphs and for further references on this topic.

Convention 2.2. Let X be a connected graph of bounded degree with the simplicial metric d. For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S.

If S is a finite subset of the vertex set of X and $k \geq 1$ is an integer, we will denote by $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$ the set of all vertices v of X such that $d_X(v, S) \leq k$. Also, we will denote $\eth^X S = \eth S := \mathcal{N}_1(S) - S$.

The number

$$\iota(X) := \inf\{\frac{|\eth S|}{|S|} \ : \ S \text{ is a finite nonempty subset of the vertex set of } X\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of X.

There are many alternative definitions of non-amenability:

Proposition 2.3. *Let X be a connected graph of bounded degree with simplicial metric d. Then the following conditions are equivalent:*

(1) *The graph X is non-amenable.*
(2) (Følner criterion) *We have $\iota(X) > 0$.*
(3) (Gromov's Doubling Condition) *There is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq 2|S|.$$
(4) *For any integer $q > 1$ there is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq q|S|.$$
(5) *For some $0 < \sigma < 0$ $p^{(n)}(x, y) = o(\sigma^n)$ for any $x, y \in VX$.*
(6) *The pseudogroup $W(X)$ consisting of all bijections between subsets of VX which are "bounded perturbations of the identity" admits a "paradoxical decomposition" (see* [16] *for definitions).*
(7) ("Grasshoper criterion") *There exists a map $\phi : VX \to VX$ such that $\sup_{x\in VX} d(x, \phi(x)) < \infty$ and that for any $x \in VX$ $|\phi^{-1}(x)| \geq 2$.*
(8) *There exists a map $\phi : VX \to VX$ such that $\sup_{x\in VX} d(x, \phi(x)) < \infty$ and that for any $x \in VX$ $|\phi^{-1}(x)| = 2$.*
(9) *The bottom of the spectrum for the combinatorial Laplacian operator on X is > 0 (see* [21] *for the precise definitions).*
(10) *We have $H_0^{uf}(X) = 0$ (see* [13] *for the precise definition of* uniformly finite *homology groups H_i^{uf}).*
(11) *We have $H_0^{(l_p)}(X) = 0$ for any $1 < p < \infty$ (see* [24] *for the precise definition of $H_i^{(l_p)}$).*

Proof. All of the above statements are well-known, but we will still provide some sample references.

The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J.Dodziuk [21].

J.Block and S.Weinberger [13] established the equivalence of (2) and (10). Finally, G.Elek [24] proved that (2) is equivalent to (11). □

In case of regular graphs one can also characterize non-amenability in terms of co-growth.

Definition 2.4. Let X be a connected graph of bounded degree with a base-vertex x_0. Let $a_n = a_n(X, x_0)$ be the number of reduced edge-paths of length n from x_0 to x_0. Let $b_n = b_n(X, x_0)$ be the number of all edge-paths of length n from x_0 to x_0. Put

$$\alpha(X) := \limsup_{n\to\infty} \sqrt[n]{a_n} \text{ and } \alpha(X) := \limsup_{n\to\infty} \sqrt[n]{b_n}$$

Then we will call $\alpha(X)$ the *co-growth rate* of X and we will call $\beta(X)$ the *non-reduced co-growth rate* of X. These definitions are well-known to be independent of the choice of x_0.

It is easy to see that for a d-regular connected graph X we have $\alpha(X) \le d-1$ and $\beta(X) \le d$. Moreover, $\rho(X) = \frac{\beta(X)}{d}$. It turns out that non-amenability of regular graphs can be characterized in terms of the co-growth rate. The following result is was originally proved by R.Grigorchuk [39] and J.Cohen [19] for Cayley graphs of finitely generated groups and by L.Bartholdi [5] for arbitrary regular graphs.

Theorem 2.5. [5] *Let X be a connected d-regular graph with $d \ge 3$. Put $\alpha = \alpha(X)$, $\beta = \beta(X)$ and $\rho = \rho(X)$. Then*

$$\rho = \frac{2\sqrt{d-1}}{d} \quad \textit{if} \quad 1 \le \alpha \le \sqrt{d-1}$$

and

$$\rho = \frac{\sqrt{d-1}}{d}\left(\frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}}\right) \quad \textit{if} \quad \sqrt{d-1} \le \alpha \le d-1.$$

In particular $\rho < 1 \iff \alpha < d-1 \iff \beta < d$.

3. Hyperbolic metric spaces

The basic information about Gromov-hyperbolic metric spaces and word-hyperbolic groups can be found in [40, 20, 32, 1, 14, 25, 4] and other sources. We will briefly recall the main definitions.

If (X, d) is a geodesic metric space and $x, y \in X$, we shall denote by $[x, y]$ a geodesic segment from x to y in X.

Definition 3.1 (Gromov product)**.** Let (X, d) be a metric space and suppose $x, y, z \in X$. We set

$$(x, y)_z := \frac{1}{2}[d(z, x) + d(z, y) - d(x, y)]$$

Note that $(x,y)_z = (y,x)_z$.

The following is one of the many equivalent definitions of hyperbolicity [1].

Definition 3.2. [1] Let (X,d) be a geodesic metric space. We say that (X,d) is *δ-hyperbolic* (where $\delta \geq 0$) if for any $p,x,y,z \in X$ we have:

$$(x,y)_p \geq \min\{(x,z)_p, (y,z)_p\} - 4\delta.$$

The space X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Definition 3.3 (Word-hyperbolic group)**.** A finitely generated group G is said to be *word-hyperbolic* if for some (and hence for any) finite generating set A of G the Cayley graph $\Gamma(G,A)$ is hyperbolic.

Definition 3.4 (Gromov product for sets)**.** Let (X,d) be a metric space. Let $x \in X$ and $Q, Q' \subseteq X$. Put $(Q,Q')_x := sup\{(q,q')_x \mid q \in Q, q' \in Q'\}$.

4. Quasiconvex subgroups of hyperbolic groups

A detailed background information on quasiconvex subgroups of hyperbolic groups can be found in [1, 20, 32, 68, 4, 31, 51, 54, 38, 34] and other sources.

Convention 4.1. Suppose G is a finitely generated group with a fixed finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with respect to A. We will denote the word-metric corresponding to A on X by d_A. Also, for $g \in G$ we will denote $|g|_A := d_A(1,g)$.

Definition 4.2 (Quasiconvexity)**.** A subset $Z \subseteq X$ is called *ϵ-quasiconvex*, where $\epsilon \geq 0$, if for any $z_1, z_2 \in Z$ and any geodesic $[z_1,z_2]$ in X the segment $[z_1,z_2]$ is contained in the closed ϵ-neighborhood of Z. A subset $Z \subseteq X$ is *quasiconvex* if it is ϵ-quasiconvex for some $\epsilon \geq 0$. A subgroup $H \leq G$ is *quasiconvex* in G with respect to A if $H \subseteq X$ is a quasiconvex subset.

It turns out [20, 32, 4, 31] that for subgroups of word-hyperbolic groups quasiconvexity is independent of the choice of a finite generating set for the ambient group. Thus a subgroup H of a hyperbolic group G is termed *quasiconvex* if $H \subseteq \Gamma(G,A)$ is quasiconvex for some finite generating set A of G.

We summarize some well-known basic facts regarding quasiconvex subgroups:

Proposition 4.3. *Let G be a word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with the word-metric d_A induced by A. Then:*

(1) [20, 32] *If $H \leq G$ is a subgroup, then either H is virtually cyclic (in which case H is called* elementary*) or H contains a free subgroup F of rank two which is quasiconvex in G (in which case H is said to be* non-elementary*).*
(2) [1, 20, 32] *Every cyclic subgroup of G is quasiconvex in G.*

(3) [1, 20, 32] *If $H \le G$ is quasiconvex then H is finitely presentable and word-hyperbolic.*
(4) [20, 32, 4, 31] *Suppose $H \le G$ is generated by a finite set Q inducing a word-metric d_Q on H. Then H is quasiconvex in G if and only if there is $C > 0$ such that for any $h_1, h_2 \in H$*

$$d_Q(h_1, h_2) \le Cd(h_1, h_2).$$

(5) [31] *The set $\mathcal{L}$ of all A-geodesic words is a regular language which provides a bi-automatic structure for G. Moreover, a subgroup $H \le G$ is quasiconvex if and only if H is $\mathcal{L}$-rational, that is the set $\mathcal{L}_H = \{w \in \mathcal{L} \,|\, \overline{w} \in H\}$ is a regular language.*
(6) [68] *If $H_1, H_2 \le G$ are quasiconvex, then $H_1 \cap H_2 \le G$ is quasiconvex.*
(7) [51] *Suppose $H \le G$ is an infinite quasiconvex subgroup. Then H has finite index in its* commensurator $Comm_G(H)$, *where*

$$Comm_G(H) :=$$
$$= \{g \in G \,|\, [H : g^{-1}Hg \cap H] < \infty \text{ and } [g^{-1}Hg : Hg^{-1}Hg \cap H] < \infty\}.$$

Part 1 of the above proposition implies that a non-elementary subgroup of a hyperbolic group is non-amenable.

5. Proof of the main result

Let G be a non-elementary word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G. Let $\delta \ge 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ-hyperbolic. Let $H \le G$ be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

The following useful fact follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]:

Lemma 5.1. *There exists an integer constant $K = K(G, H, A) > 0$ with the following properties.*

Suppose $g \in G$ is shortest with respect to d in the coset class Hg. Let $h \in H$ be an arbitrary element. Then $(g, h)_1 \le K$ (and hence $(g, H)_1 \le K$).

Lemma 5.2. *Suppose $g \in G$ such that $(g, H)_1 \le T_1$ and $|g|_A > T_1 + T_2 + \delta$ where $T_1, T_2 > 0$. Suppose $f \in G$ is such that $|f|_A \le T_2$. Then $(gf, H)_1 \le T_1 + \delta$.*

Proof. Note that $|g|_A = (g, gf)_1 + (1, gf)_g$. Since $(1, gf)_g \le d(g, gf) = |f|_A \le T_2$, we conclude that $(g, gf)_1 = |g|_A - (1, gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta$. Therefore for any $h \in H$ we have

$$T_1 + \delta \ge (g, h)_1 + \delta \ge \min\{(g, gf)_1, (gf, h)_1\}$$

and hence $(gf, h)_1 \le T_1 + \delta$ since $(g, gf)_1 > T_1 + \delta$. Since $h \in H$ was arbitrary, this means that $(gf, H)_1) \le T_1 + \delta$, as required. □

Lemma 5.3. *Suppose $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$. Then there is $h \in H$ such that $hg_1 = g_2$ and that*

$$|h|_A \leq (g_1, H)_1 + (g_2, H)_1.$$

Proof. Since $Hg_1 = Hg_2$, there is $h \in H$ with $hg_1 = g_2$. Hence

$$|h|_A = (h, g_2)_1 + (1, hg_1)_h = (h, g_2)_1 + (h^{-1}, g_1)_1 \leq (g_2, H)_1 + (g_1, H)_1,$$

as required. □

Proof of Theorem 1.2. Let $K = K(G, H, A) > 0$ be the constant provided by Lemma 5.1. Put $Y = \Gamma(G, H, A)$. Thus Y is a connected $2m$-regular infinite graph where m is the number of elements in A. We denote the simplicial metric on Y by d_Y.

Let N be the number of all elements $g \in G$ with $|g|_A \leq 2K + 2\delta$. In particular this means that Y has at most N vertices within the distance $2K + 2\delta$ of $H1 \in VY$.

Since G is non-elementary word-hyperbolic and thus non-amenable, the Cayley graph $X = \Gamma(G, A)$ is non-amenable. Hence by part 4 of Proposition 2.3 there is a constant $k' > 0$ such that for any finite nonempty subset S of G the k'-neighborhood of S in X has at least $4N|S|$ vertices. Choose $k'' > 1$ such that for any vertex $Hg \in VY$ with $d_Y(H1, Hg) \leq K+\delta+k'$ the k''-neighborhood of Hg has at least $4N_1$ vertices, where N_1 is the number of elements of G of length at most $K+\delta+k'$. Such k'' exists since by assumption $[G : H] = \infty$ and hence the graph Y is infinite. Put $k := \max\{k', k''\}$.

Suppose now that $F \subset VY$ is a finite non-empty subset. Write $F = F_1 \sqcup F_2$ where F_1 is the intersection of F with the closed ball of radius $K + \delta + k'$ in Y.

If $|F_1| \geq |F|/2$ then $|F| \leq 2N_1$ and the k-neighborhood of F in Y has at least $4N_1 \geq 2|F|$ vertices. Suppose now that $|F_1| < |F|/2$, so that $|F_2| \geq |F|/2$.

Thus

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where $|F_2| = t$ and where each $g_i \in G$ is shortest in Hg_i with $|g_i|_A > K + \delta + k'$. By Lemma 5.1 $(g_i, H)_1 \leq K$. Hence by Lemma 5.2 for any $f \in G$ with $|f|_A \leq k'$ and for each $i = 1, \dots, t$ we have $(g_i f, H)_1 \leq K + \delta$.

Let $S := \{g_1, \dots, g_t\}$ and let S' be the set of all vertices of X contained in the k'-neighborhood of S in X. By the choice of k' we have $|S'| \geq 4N|S| = 4N|F_2|$. On the other hand Lemma 5.3 implies that if $g, g' \in S'$ are such that $Hg = Hg'$ then $hg = g'$ for some $h \in H$ with $|h|_A \leq 2K + 2\delta$. By the choice of N this means that the set $F' := \{Hg, |g \in S'\}$ contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \geq 2|F|$$

distinct elements. However, F' is obviously contained in the k-neighborhood of F in Y.

Thus we have verified that for any finite non-empty subset $F \subseteq VY$ the k-neighborhood of F in Y contains at least $2|F|$ vertices. By the Doubling

Condition (part 3 of Proposition 2.3) this means that Y is non-amenable, as required. □

We can now obtain Corollary 1.4 from the Introduction.

Corollary 5.4. *Let $G = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_m\rangle$ be a non-elementary hyperbolic group and let $H \le G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n representing elements of H. Let b_n be the number of all words in A of length n representing elements of H.*

Then

$$\limsup_{n\to\infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n\to\infty} \sqrt[n]{b_n} < 2k.$$

Proof. Note that $k \ge 2$ since G is non-elementary. Put $A = \{x_1, \dots, x_k\}$ and $Y = \Gamma(G, H, A)$. We choose $x_0 := H1 \in VY$ as the base-vertex of Y. Note that Y is 2k-regular by construction. Also, for any vertex x of Y and any word w in $A \cup A^{-1}$ there is a unique path in Y with label w and origin x. The definition of Schreier subgroup graphs also implies that:
(1) A freely reduced word w represents an element of H if and only if the path in Y labeled w with origin x_0 terminates at x_0.
(2) A word w represents an element of H if and only if the path in Y labeled w with origin x_0 terminates at x_0.

Therefore $a_n(Y)$ equals the number of freely reduced words in the alphabet $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n representing elements of H. Similarly, $b_n(Y)$ equals the number of all words in A of length n representing elements of H.

By Theorem 1.2 Y is non-amenable. Hence by Theorem 2.5 $\alpha(Y) < 2k-1$ and $\beta(Y) < 2k$, as required. □

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: kapovich@math.uiuc.edu