A NON-QUASICONVEXITY EMBEDDING THEOREM FOR HYPERBOLIC GROUPS.

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Abstract. We show that if G is a non-elementary torsion-free word hyperbolic group then there exists another word hyperbolic group G^*, such that G is a subgroup of G^* but G is not quasiconvex in G^*. We also prove that any non-elementary subgroup of a torsion-free word hyperbolic group G contains a free group of rank two which is malnormal and quasiconvex in G.

1.Introduction.

Word hyperbolic groups introduced by M.Gromov in [13] have played a very important role in the recent progress of geometric and combinatorial group theory. It has turned out that many groups arising in a traditionally geometric context, such as fundamental groups of closed compact manifolds admitting a metric of strictly negative curvature, have the property of being word hyperbolic. On the other hand, most finitely presented groups (in a certain probabilistic sense), also belong to the class of word hyperbolic groups. Moreover, word hyperbolic groups possess a number of very good algorithmic and combinatorial properties which do not hold for the class of finitely presented groups in general. For example, the word problem and the conjugacy problem are solvable in any word hyperbolic group [13, Theorem 7.4.B]. Even more surprisingly, the isomorphism problem is solvable in the class of torsion-free word hyperbolic groups [28]. We will provide some elementary facts about hyperbolic groups in Section 2. For a more detailed discussion the reader is referred to [13], [1], [11],[14].

A particularly important and interesting class of subgroups of word hyperbolic groups are the so-called *quasiconvex subgroups*. Roughly speaking, a finitely generated subgroup H of a word hyperbolic group G is quasiconvex in G if for any word metric d_G on G and any word metric d_H on H the inclusion map $i : (H, d_H) \longrightarrow (G, d_G)$ is a bi-Lipschitz embedding. (see Section 2 for a more careful definition).

Quasiconvex subgroups of word hyperbolic groups have many good properties. For example, a quasiconvex subgroup of a word hyperbolic group is always finitely presented and itself word hyperbolic. The intersection of any two quasiconvex subgroups is always quasiconvex (and so is finitely generated and finitely presented). Quasiconvex subgroups are of particularly great importance when one studies amalgamated free products and HNN-extensions of hyperbolic groups [13], [4], [18], [6], [12], [25].

From a geometric standpoint, quasiconvex subgroups of word hyperbolic groups are closely related to geometrically finite subgroups of classical hyperbolic groups. More precisely, suppose G is a geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics. Then G is word hyperbolic and a subgroup H of G is quasiconvex in G if and only if H is geometrically finite [32]. Some of the properties of quasiconvex subgroups will be given in Section 2. For a more detailed discussion on quasiconvexity the reader is referred to [13], [14], [19], [22].

One of the most fundamental results pertaining to quasiconvexity is the following theorem of M.Gromov [13, Theorem 8.1.D] (see also [1, Proposition 3.2]).

Proposition 1.1. *Let G be a word hyperbolic group and let C be an infinite cyclic subgroup of G. Then C is quasiconvex in G.*

Thus for an infinite cyclic group quasiconvexity is an absolute property and does not depend on the embedding in an ambient hyperbolic group. This leads us to the following definition.

1991 *Mathematics Subject Classification*. Primary 20F10; Secondary 20F32.

Definition 1.2. Let G be a word hyperbolic group. We say that G is *absolutely quasiconvex* if for any word hyperbolic group G^*, containing G as a subgroup, G is quasiconvex in G^*.

It is natural to ask, then, whether there are any absolutely quasiconvex groups other than the infinite cyclic group. Perhaps surprisingly, it turns out that the answer to this question is no, at least if we restrict ourselves to the class of torsion-free word hyperbolic groups. The main result of the present paper is the following

Theorem A. *Let G be a non-elementary (that is not virtually cyclic) torsion-free word hyperbolic group. Then there exists a word hyperbolic group G^* such that G is a subgroup of G^* and G is not quasiconvex in G^*.*

Since any non-trivial torsion-free virtually cyclic group is in fact infinite cyclic, this result immediately implies

Theorem B. *Let G be a non-trivial torsion-free absolutely quasiconvex group. Then G is infinite cyclic.*

We also prove the following statement which appears to be of independent interest.

Theorem C. *Let G be a torsion-free word hyperbolic groups and let Γ be a non-elementary (i.e. not cyclic) subgroup of G. Then there exists a subgroup H of Γ such that H is a free group of rank two which is quasiconvex and malnormal in G. (Recall that a subgroup H of G is called malnormal if for any $g \in G - H$ we have $H \cap g^{-1}Hg = 1$.)*

We would like to note that up to this point there were very few known examples of finitely generated subgroups of word hyperbolic groups that are not quasiconvex.

The first example of this kind is provided in a remarkable work of E.Rips [27]. Given any finitely presented group Q, Rips constructs a finitely generated small cancellation $C(7)$-group G and a short exact sequence

$$1 \to K \to G \to Q \to 1$$

where K is a two-generated group. The group G is word hyperbolic by the basic results of small cancellation theory. It was shown in [1] that if a normal subgroup of a word hyperbolic group is quasiconvex then it is either finite or has finite index. Suppose now that in the Rips' construction the group Q is chosen to be a non-hyperbolic infinite group (e. g. $\mathbb{Z} \times \mathbb{Z}$). Since G is word hyperbolic but G/K is not, we conclude that K is infinite. Thus K is an infinite two-generated subgroup of infinite index in G which is normal in G. Therefore K is not quasiconvex.Therefore in Rips's example H is not quasiconvex. It was later noticed in [30] that K is not even finitely presentable.

If M is a 3-manifold obtained from a cylinder over a closed oriented surface S of genus at least two by gluing upper and lower boundaries of this cylinder along a pseudo-anosov homeomorphism of S then the fundamental group G of the resulting manifold M is word hyperbolic and it contains a subgroup H isomorphic to the fundamental group of S which is not quasiconvex in G. This follows from the result of Thurston [33] which asserts that in this situation M admits a metric of constant negative curvature. Therefore (see [13]) G is word hyperbolic. Besides M fibers over a circle with a fiber S and thus there is a short exact sequence $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$. Hence by the above mentioned result of [1] $\pi_1(S)$ is not quasiconvex in G.

Later M.Bestvina and M.Feign [4] showed that there is an analog of the previous example for free groups, that is there is an HNN-extension of a free group by a suitable automorphism which is word hyperbolic. Indeed, their result shows that if F is a finitely generated noncyclic free group then there is a word hyperbolic group G and a short exact sequence $1 \to F \to G \to \mathbb{Z} \to 1$ and, of course, F is not quasiconvex in G.

The existence of a hyperbolic 3-manifold fibering over a circle (see the example of W.Thurston above) provided a basis for constructing other examples of nonquasiconvex subgroups of hyperbolic groups. In [7] G.Mess and B.Bowditch showed that there is discrete cocompact group G of isometries of $\mathbb{H}^4$ containing a subgroup H which is finitely generated but not finitely presentable and therefore not quasiconvex. L.Potyagailo constructed in [26] a geometrically finite subgroup G of $SO(4,1)$ without parabolics which contains a finitely generated subgroup H which is not finitely presentable, contains infinitely many conjugacy classes of finite subgroups and such that $G/H = \mathbb{Z}$; (clearly this H is not quasiconvex).

Later N.Brady [3] constructed an example of a word hyperbolic group G which possesses a finitely presented subgroup H such that H itself is not word hyperbolic. As we mentioned before, quasiconvex subgroups of word hyperbolic groups are word hyperbolic. Therefore the subgroup H is obviously not quasiconvex in G.

Thus Theorem A provides a large class of new examples of finitely presented non-quasiconvex subgroups of hyperbolic groups. Theorem B shows that there are no rigid torsion-free groups (except for cyclic groups) in the sense of always being quasiconvex.

We already mentioned the parallel between quasiconvexity and geometric finiteness. It is interesting to note that there are also very few examples of non-geometrically finite groups acting discretely on $\mathbb{H}^n$. There is some reason to believe that, in contrast to Theorem B, there may exist some "absolutely geometrically finite" groups that are not virtually cyclic. Consider, for example, a group G which is the fundamental group of a closed hyperbolic 3-manifold fibering over a circle. It would be interesting to investigate whether G admits a discrete non-geometrically finite action in some higher-dimensional hyperbolic space $\mathbb{H}^n$, $n > 3$.

The paper is organized as follows. In section 3 we discuss the Combination Theorem of M.Bestvina and M.Feighn. Partially our motivation for writing this paper was to give an algebraic version of this useful result and demonstrate its various applications in a purely group theoretic context (the original statement and the proof were given in topological terms). In section 4 we give the proof of the main results using Theorem C and the Combination Theorem. Theorem C itself is proved in section 6 by reducing it to a certain technical statement (Theorem 5.16) about free groups. The proof of Theorem 5.16 is the subject of section 5.

It is possible, as indeed was the original intension of the author, to give a more geometric proof of Theorem C relying on small cancellation methods in hyperbolic groups in the spirit of [10], [8] and [24]. However, it turns out that complete and accurate proofs using this approach are quite lengthy and cumbersome (especially the existential part of Theorem C rather than just a set of sufficient conditions for several elements to generate a free, quasiconvex and malnormal subgroup). Therefore the author chose a more algebraic way of proving Theorem C presented in sections 5 and 6 which is also easier to make rigorous, precise, complete and correct.

2.Preliminary facts and definitions.

Definition 2.1. If (X, d) is a metric space, then a map $f\colon [a, b] \to X$ is called a *geodesic segment* if for any $[a_1, b_1] \subset [a, b]$

$$|a_1 - b_1| = d(f(a_1), f(b_1)).$$

(Sometimes, by abuse of notation, we will identify such a map f with its image and denote it by $[f(a), f(b)]$.)

A metric space (X, d) is said to be *geodesic* if any two points in X can be joined by a geodesic segment.

A geodesic space (X, d) is called *δ-hyperbolic* if for each triangle Δ with geodesic sides in X and for any point x on one of the sides of Δ there is a point y on one of the two other sides such that $d(x, y) \le \delta$.

If G is a group and $X = S \cup S^{-1}$ is a finite generating set for G then a *Cayley graph* $\Gamma(G, X)$ of G is an oriented labeled graph with $\{g | g \in G\}$ as a set of vertices and an oriented edge $e = (g, ga)$ labeled by a for each $g \in G$, $a \in X$. It is not hard to see that $\Gamma(G, X)$ is a connected locally finite graph. If we put each edge to be isometric to a unit interval, we can define the length of an edge-path in $\Gamma(G, X)$. Now for any vertices g, h of $\Gamma(G, X)$ put $d_X(g, h)$ to be the minimal length of an edge-path connecting g to h. Then d_X is a metric on G which can be naturally extended to a metric (which we also denote d_X) on $\Gamma(G, X)$. The metric d_X is called a *word metric* corresponding to X. It is easy to see that $(\Gamma(G, X), d_X)$ is a geodesic metric space. We denote the set of all words over X (that is the free monoid on X) by X^* and the element of G represented by a word $w \in X^*$ by $\overline{w}$. For every $g \in G$ we denote $l_X(g) = d_X(g, 1)$ and call $l_X(g)$ the *word length* of G with respect to X. Note that for any $g \in G$ the word length $l_X(g)$ is the minimal number $n \ge 0$ such that g can be expressed as a product

$$g = x_1 \dots x_n,$$

where all $x_i \in X$.

Proposition-Definition 2.2. (see [1, Section 2] for proof) Let G be a finitely generated group. Then the following conditions are equivalent:

(1) for some finite generating set $X = S \cup S^{-1}$ of G and for some $\delta \geq 0$ the Cayley graph $\Gamma(G, X)$ is δ-hyperbolic;
(2) for any finite generating set $X = S \cup S^{-1}$ of G there is $\delta \geq 0$ such that the Cayley graph $\Gamma(G, X)$ is δ-hyperbolic.

If any of these conditions is satisfied, the group G is called *word hyperbolic*.

We refer the reader to [13], [1] and [14] for more information about word hyperbolic groups and their properties.

Let G be a finitely generating group with a finite generating set $X = S \cup S^{-1}$. Then any word w over X corresponds to a path $p(w)$ in $\Gamma(G, X)$ from 1 to $\overline{w}$ of length $l(w)$. We say that w is *geodesic* with respect to d_X if the path $p(w)$ is geodesic in $\Gamma(G, X)$, that is if $l(w) = l_X(\overline{w})$.

In general, a subset Y of a geodesic metric space (X, d) is termed ϵ-*quasiconvex* in X if for any $y_1, y_2 \in Y$ any geodesic $[y_1, y_2]$ in X, joining points y_1 and y_2, is contained in the ϵ-neighborhood of Y.

Proposition-Definition 2.3. (see [1] and [6] for proof) Let G be a word hyperbolic group and A be a subgroup of G. Then the following conditions are equivalent:

(1) for **some** finite generating set $X = S \cup S^{-1}$ of G there is an $\epsilon \geq 0$ such that A is an ϵ-quasiconvex subset of $\Gamma(G, X)$;
(2) for **any** finite generating set $X = S \cup S^{-1}$ of G there is an $\epsilon \geq 0$ such that A is an ϵ-quasiconvex subset of $\Gamma(G, X)$;
(3) the group A is finitely generated and for **some** finite generating set $Y = T \cup T^{-1}$ of A and for **some** finite generating set $X = S \cup S^{-1}$ of G there is a constant $C > 0$ such that for any $a \in A$
$$d_X(a, 1)/C \leq d_Y(a, 1) \leq C d_X(a, 1);$$
(4) the group A is finitely generated and for **any** finite generating set $Y = T \cup T^{-1}$ of A and for **any** finite generating set $X = S \cup S^{-1}$ of G there is a constant $C > 0$ such that for any $a \in A$
$$d_X(a, 1)/C \leq d_Y(a, 1) \leq C d_X(a, 1).$$

If any of these conditions is satisfied then A is called a *quasicovex subgroup* of G.

The reader is refered to [13],[1], [16], [22], [19] for the summary of the properties of quasiconvex subgroups of word hyperbolic groups.

3.The combination theorem.

Suppose $G = \pi_1(\mathbb{A}, T)$ where $\mathbb{A}$ is a graph of groups (see [2] and [29] for definition and properties of graphs of groups) with underlying finite graph A, maximal subtree T and such that every vertex group is word hyperbolic and every edge monomorphism is a quasi-isometric embedding (that is the image of an edge group under the edge monomorphism is quasiconvex in the appropriate vertex group).

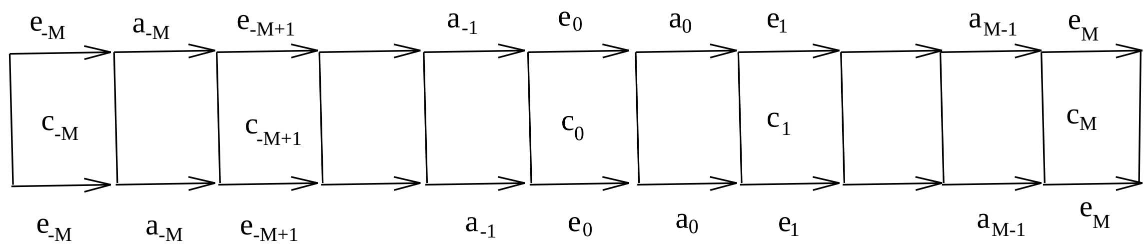
Question. When is G word hyperbolic?

Partial results in this direction can be found in [13], [6], [18], [12] and some other papers. However, the most complete answer to this question up to date is given in the Combination Theorem of M.Bestvina and M.Feighn [4], [5]. Before formulating their result we need to introduce some definitions.

Definition 3.1 (Annulus). Let $\mathbb{A}$ be a graph of groups with vertex groups A_v, $v \in VA$ and edge groups A_e, $e \in EA$. For an edge $e \in EA$ denote the edge homomorphisms by $\alpha_e : A_e \longrightarrow A_{\partial_0(e)}$ and $\alpha_{\overline{e}} : A_e \longrightarrow A_{\partial_1(e)}$.

A *combinatorial annulus* of length $2M + 1$ is a diagram Σ as in Figure 3.1
satisfying the following properties:

(1) the sequence $(e_{-M}, e_{-M+1}, \ldots, e_0, \ldots, e_M)$ is an edge-path in A;



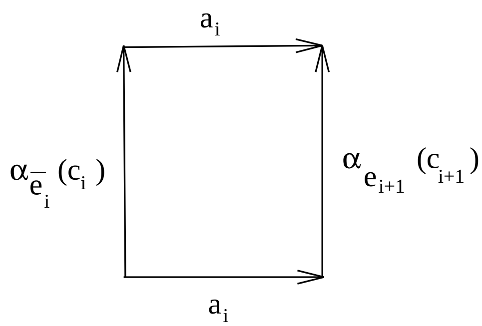


FIGURE 3.1 (ANNULUS)

(2) for every $i = -M, \dots M-1$ we have
$$a_i \in A_{v_i}$$
where $v_i = \partial_1(e_i) = \partial_0(e_{i+1})$;

(3) for every $i = -M, \dots, M$ we have
$$c_i \in A_{e_i}, c_i \neq 1$$
and
$$a_i{}^{-1}\alpha_{\overline{e_i}}(c_i)a_i = \alpha_{e_{i+1}}(c_{i+1})$$

Thus in G

$$(e_{-M}a_{-M}e_{-M+1}\dots a_{M-1}e_M)^{-1} \cdot \alpha_{e_{-M}}(c_{-M}) \cdot e_{-M}a_{-M}e_{-M+1}\dots a_{M-1}e_M = \alpha_{\overline{e}_M}(c_M)$$

and

$$\alpha_{e_{-M}}(c_{-M}) \cdot e_{-M}a_{-M}e_{-M+1}\dots a_{M-1}e_M = \\ = e_{-M}a_{-M}e_{-M+1}\dots a_{M-1}e_M \cdot \alpha_{\overline{e}_M}(c_M)$$

Definition 3.2 (Essential annulus). An annulus Σ as above is called *essential* if the sequence

$$(e_{-M}, a_{-M}, e_{-M+1}, \dots, a_{M-1}, e_M)$$

contains no "pinches" that is it has no subsequences of the form

$$e, \alpha_{\overline{e}}(s), \overline{e}$$

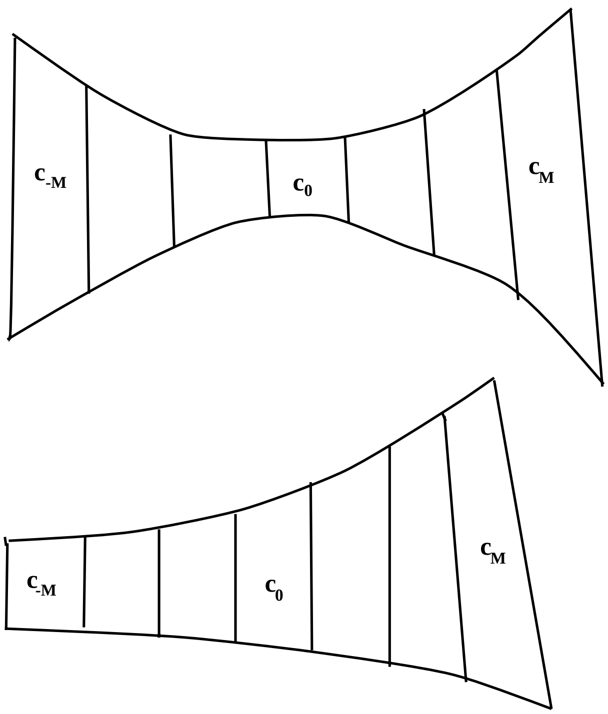


FIGURE 3.2 (HYPERBOLIC ANNULI)

where $s \in A_e$, and e is an edge of A.

Remark 3.3. From the geometric viewpoint if Σ is an essential annulus then the sequence $(e_{-M}, a_{-M}, e_{-M+1}, \dots, a_{M-1}, e_M)$ represents an edge-path without backtracks of length $2M+1$ in the Bass-Serre universal covering tree $\hat{T}$ corresponding to the graph of groups $\mathbb{A}$. Moreover, the definition of an annulus in this case is just an algebraic restatement of the fact that there is a nontrivial element in the fundamental group $\pi_1(\mathbb{A}, *)$ of the graph of groups $\mathbb{A}$ which fixes this edge-path pointwise.

Definition 3.4. Let $\mathbb{A}$ be a graph of groups as above. Fix generating sets for all the vertex groups A_v, all the edge groups A_e and the word metrics d_v, d_e induced by them. Then if Σ is an annulus as in Definition 3.1, we say that

(1) the *girth* of Σ is $d_{e_0}(c_0)$;
(2) the *width* of Σ is $\max\{d_{v_i}(a_i) | -M \le i < M\}$.

Let $\lambda > 1$. An annulus Σ is called λ-*hyperbolic* (see Figure 3.2) if

$$\lambda d_{e_0}(c_0) \le \max\{d_{e_M}(c_M), d_{e_{-M}}(c_{-M})\}.$$

Theorem 3.5 (Combination Theorem). *(M.Bestvina and M.Feighn* [5]*) Suppose $G = \pi_1(\mathbb{A}, T)$ where $\mathbb{A}$ is a graph of groups with underlying finite graph A, maximal subtree T and such that every vertex group is word hyperbolic. Suppose also that every edge monomorphism $\alpha_e : A_e \longrightarrow A_{\partial_0(e)}$ is a quasi-isometric embedding (that is $\alpha_e(A_e)$ is quasiconvex in $A_{\partial_0(e)}$). Fix finite generating sets for all A_v, A_e and the word metrics d_v, d_e induced by them.*

Suppose there exist $\lambda > 1$ and $M \ge 1$ such that the following holds. For each $\rho > 0$ there is $H(\rho)$ such that any essential annulus of length $2M+1$, width at most ρ and girth at least $H(\rho)$ is λ-hyperbolic.

Then G is word hyperbolic.

Remark 3.6. Suppose that the conditions of the Combination Theorem as stated above are satisfied for a particular choice of generating sets for the vertex groups A_v and edge groups A_e. Then it is obvious from

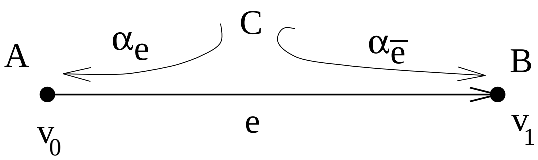


FIGURE 3.3

the definition of an annulus that for any integer $k > 0$ every essential annulus of width at most ρ and girth at least $H(\rho)$ and of length $2kM + 1$ is λ^k-hyperbolic. Therefore the conditions of the Combination Theorem for the graph of groups $\mathbb{A}$ are satisfied (with different M and λ) for any other choice of finite generating sets for the vertex groups and the edge groups of $\mathbb{A}$.

There is one easy corollary of the Combination Theorem that is extremely useful when working with hyperbolic groups.

Corollary 3.7. *Let $\mathbb{A}$ be a finite graph of groups such that all vertex groups are word hyperbolic and the images of the edge groups under edge monomorphisms are quasiconvex in the appropriate vertex groups. Let T be a maximal tree in the underlying graph A of $\mathbb{A}$ and let $G = \pi_1(\mathbb{A}, T)$ be the fundamental group of the graph of groups $\mathbb{A}$ with respect to T.*

Suppose there is an integer $M > 0$ such that there are no essential annuli of length $2M + 1$ corresponding to $\mathbb{A}$.

Then G is word hyperbolic.

A direct restatement of Corollary 3.7 yields the following.

Corollary 3.8. *Let $\mathbb{A}$ be a finite graph of groups such that all vertex groups are word hyperbolic and the images of the edge groups under edge monomorphisms are quasiconvex in the appropriate vertex groups. Let T be a maximal tree in the underlying graph A of $\mathbb{A}$ and let $G = \pi_1(\mathbb{A}, T)$ be the fundamental group of the graph of groups $\mathbb{A}$ with respect to T. Let $\hat{T}$ be the Bass-Serre universal covering tree of $\mathbb{A}$ on which G acts.*

Suppose there is an integer $M > 0$ such that no nontrivial element of G fixes a segment of length M in $\hat{T}$ pointwise.

Then G is word hyperbolic.

We would like to point out two explicit and most often used applications of the above results. First recall the following definition.

Definition 3.9. A subgroup H of a group G is called *malnormal* if for every $g \in G$, $g \notin H$

$$gHg^{-1} \cap H = 1$$

Proposition 3.10. *Let A and B be word hyperbolic groups such that the subgroup $C = A \cap B$ is quasiconvex in both A and B. Suppose also that C is malnormal in B. Then the amalgamated free product $G = A *_C B$ is word hyperbolic.*

Proof. Let $\mathbb{Y}$ be the edge of groups with the vertex groups A, B and the edge group C as shown in Figure 3.3.

Thus Y has the edge e with the endpoints $\partial_0(e) = v_0$, $\partial_1(e) = v_1$. The vertex groups for $\mathbb{Y}$ are $Y_{v_0} = A$, $Y_{v_1} = B$ and the edge groups are $Y_e = Y_{\overline{e}} = C$. The boundary monomorphisms $\alpha_e : C \to A$ and $\alpha_{\overline{e}} : C \to B$ are the inclusions. Then obviously $G = A *_C B \cong \pi_1(\mathbb{Y}, v_0)$.

We claim that there are no essential annuli of length 3 for the graph of groups $\mathbb{Y}$. Indeed, suppose that Σ is an essential annulus of length 3. Then up to axial symmetry Σ has the form as in Figure 3.4, (that is the sequence of edges in the top label of Σ is $\overline{e}, e, \overline{e}$ rather than $e, \overline{e}, e$).

Then $c_{-1}, c_0, c_1 \in C - \{1\}$, $a_{-1} \in A$, $b_0 \in B$. Since the boundary monomorphisms are just inclusions, by the definition of an annulus we have

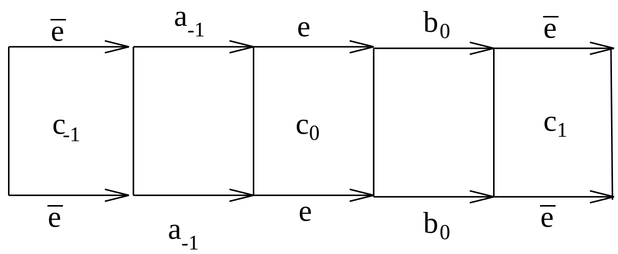


Figure 3.4

$$b_0^{-1}c_0b_0 = c_1 \text{ in } B.$$

Since C is malnormal in B and $c_0 \neq 1$, this implies $b_0 \in C$. Therefore the sequence $e, b_0, \overline{e}$ is a "pinch" and the annulus Σ is not essential. This contradicts our assumptions.

Thus there are indeed no essential annuli of length 3 and therefore the group G is word hyperbolic by Corollary 3.7.

Proposition 3.11.
Let G be word hyperbolic group. Let C_1 and C_2 be isomorphic quasiconvex subgroups of G and let $\phi : C_1 \to C_2$ be an isomorphism. Suppose that both C_1 and C_2 are malnormal on G and, moreover, no nontrivial element of C_1 is conjugate in G to an element of C_2.

Then the HNN-extension

$$K =< G, t | t^{-1}ct = \phi(c), c \in C_1 >$$

is word hyperbolic.

Proof. Once again, it is easy to see that if $\mathbb{A}$ is the loop of groups associated to the HNN presentation of K then there are no essential annuli of length 3 corresponding to $\mathbb{A}$ (the details are left to the reader). Therefore by Corollary 3.7 the group K is word hyperbolic.

Remark 3.12. Suppose that $\mathbb{A}$ is a finite graph of groups such that all the edge groups are infinite cyclic. Let T be a maximal tree in A and let $G = \pi_1(\mathbb{A}, T)$. It is not hard to see that if $\mathbb{A}$ has essential annuli of an arbitrary big length, then G contains a so-called Baumslag-Solitar subgroup, that is a subgroup of the form

$$B(m, n) =< a, t | t^{-1}a^n t = a^m >$$

for some $m \neq 0, n \neq 0$. Baumslag-Solitar groups can never be subgroups of word hyperbolic groups (see [16]). Therefore if $\mathbb{A}$ is a finite graph of groups with word hyperbolic vertex groups and infinite cyclic edge groups then the fundamental group G of the graph of groups $\mathbb{A}$ is word hyperbolic if and only if G does not contain Baumslag-Solitar subgroups [5].

We will later need the following simple lemma.

Lemma 3.13. *Let $\mathbb{A}$ be a graph of groups with and underlying graph A. Suppose e is an edge of A such that the subgroup $\alpha_e(A_e)$ is malnormal in the vertex group A_v where $v = \partial_0(e)$. Then there are no essential annuli Σ corresponding to $\mathbb{A}$ such that the top label of Σ contains a subsequence $\overline{e}, a, e$ where $a \in A_v$.*

Proof. Suppose, on the contrary, such an essential annulus Σ exists. Then Σ contains a sub-annulus of the form shown in Figure 3.5.

Here $a \in A_v$, $c, c' \in A_e = A_{\overline{e}}$, $c \neq 1, c' \neq 1$. Then by the definition of an annulus

$$a^{-1}\alpha_e(c)a = \alpha_e(c').$$

Since $\alpha_e(A_e)$ is malnormal in A_v, this implies that $a \in \alpha_e(A_e)$. However, this contradicts our assumption that the annulus Σ is essential.

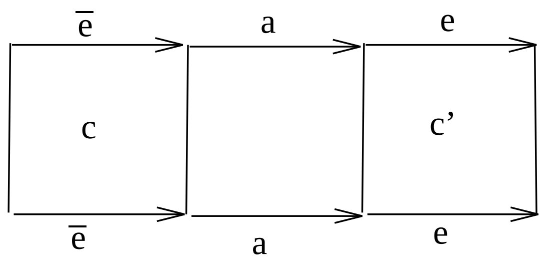


FIGURE 3.5

4.PROOFS OF THE MAIN RESULTS.

Theorem 4.1 (Theorem A). *Let G be a non-elementary torsion-free word hyperbolic group. Then there exists another word hyperbolic group G^* such that G is a subgroup of G^* and G is not quasiconvex in G^*.*

Proof of Theorem A.

Let G be as in Theorem 4.1. Then by Theorem 6.7 (which is proved later in section 6) there exists a subgroup F of G such that

(1) F is free of rank 2
(2) F is quasiconvex in G
(3) F is malnormal in G.

Say $F = F(a, b)$ is free on $a, b \in G$. For any $f \in F$ we will denote by $l_F(f)$ the length of the freely reduced word in a, b representing f.

Take $\phi : F(a, b) \longrightarrow F(a, b)$ to be an endomorphism such that

(a) the subgroup $im(\phi) = \phi(F(a, b))$ is malnormal in $F(a, b)$ (and so in G);
(b) the map ϕ is *uniformly length-expanding* that is for every $f \in F(a, b)$

$$l_F(\phi(f)) \geq 2l_F(f).$$

(Note that property (b) implies that ϕ is a monomorphism).

Such a ϕ always exists. For example, we can take $\phi(a) = ab^3a$ and $\phi(b) = ba^3b$. Note that the word ab^3a begins with a and ends with a and the word ba^3b begins with b and ends with b. Therefore for each $f \in F(a, b)$ we have $l_F(\phi(x)) = 5l_F(f)$. Also the subgroup $gp(ab^3a, ba^3b)$ is malnormal in $F(a, b)$ by Lemma 5.4.

So assume that ϕ is any endomorphism of $F(a, b)$ that satisfies conditions (a), (b) above.

Put

$$G^* =< G, t|t^{-1}at = \phi(a), t^{-1}bt = \phi(b) >$$

to be the HNN-extension of G along ϕ.

Claim Then

(1) G^* is word hyperbolic;
(2) G is not quasiconvex in G^*.

To see that G^* is word hyperbolic, note that G is isomorphic to the fundamental group of the graph $\mathbb{A}$ of groups shown in Figure 4.6.

More precisely, the underlying graph A of $\mathbb{A}$ has a single vertex v and an edge e with $\partial_0(e) = \partial_1(e) = v$. Also the vertex and edge groups are $A_v = G$, $A_e = F(x, y)$ where $F(x, y)$ is a free group of rank 2 with basis x, y. The boundary monomorphisms for $\mathbb{A}$ are defined as follows:

$$\alpha_e(x) = a, \alpha_e(y) = b$$

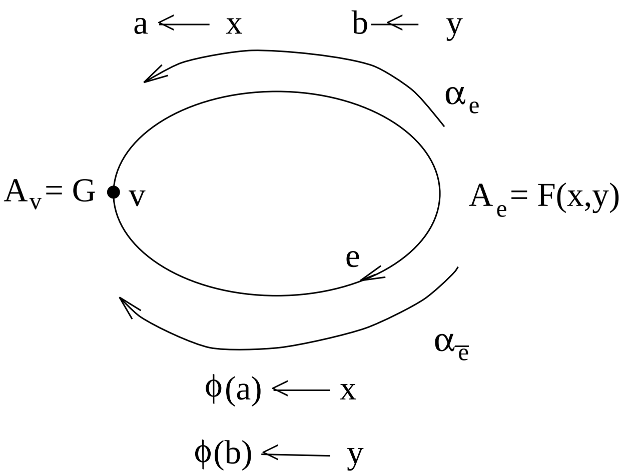


FIGURE 4.6

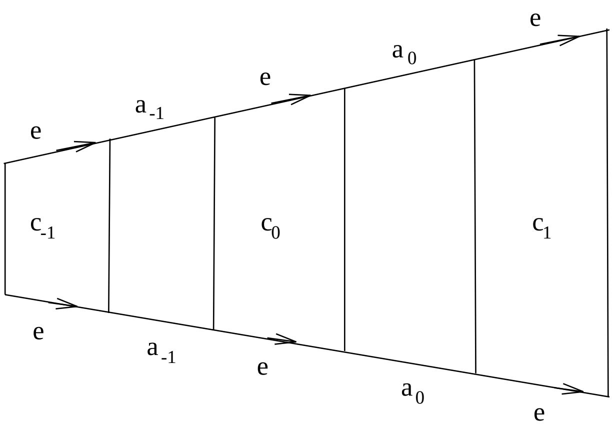


FIGURE 4.7

and

$$\alpha_{\overline{e}}(x) = \phi(a), \alpha_{\overline{e}}(y) = \phi(b).$$

Then obviously $G^* \cong \pi_1(\mathbb{A}, v)$. We will use the combination theorem to show that G^* is word hyperbolic. Note first that every essential annulus Σ is *one-directed*, that is the label of its upper boundary does not contain subsequences of the type

$$e, a, \overline{e} \quad \text{or} \quad \overline{e}, a, e$$

This follows by Lemma 3.13 because subgroups $F(a, b)$ and $\phi(F(a, b))$ are malnormal in G and the annulus Σ is essential.

Fix a finite generating set $\mathcal{G}$ for G and the word metric $d_{\mathcal{G}}$ corresponding to $\mathcal{G}$. Recall that for any $f \in F(a, b)$ we denote the freely reduced length of f in a, b by $l_F(f)$. Also if $c \in F(x, y) = A_e$, then we will denote the freely reduced length of c in x, y by $l_X(c)$.

Put $\lambda = \frac{3}{2}$ and $M = 1$. Now for every $\rho > 0$ let $q(\rho)$ be the maximum of l_F-lengths of those elements of $F(a, b)$ whose $\mathcal{G}$-length is at most ρ. Put $H(\rho) = 20q(\rho)$.

Let $\rho > 0$ be an arbitrary number. We claim that for the graph of groups $\mathbb{A}$ every essential annulus Σ of length 3, width at most ρ and girth at least $H(\rho)$ is λ-hyperbolic.

Indeed, since Σ is one-directed, up to an axial symmetry we can assume that Σ looks like the annulus shown in Figure 4.7. (That is the upper label of Σ is of the form e, a_{-1}, e, a_0, e.)

Here $a_{-1}, a \in G$, $c_{-1}, c_0, c \in F(x,y) - \{1\}$ and

$$a_i^{-1}\alpha_{\overline{e}}(c_i)a_i = \alpha_e(c_{i+1})$$

for $i = -1, 0$. Since $F(a,b)$ is malnormal in G and the subgroups $\alpha_e(A_e)$ and $\alpha_{\overline{e}}(A_e)$ are contained in $F(a,b)$, this implies that $a_0, a_{-1} \in F(a,b)$. Also $l_{\mathcal{G}}(a_{-1}), l_{\mathcal{G}}(a_0) \le \rho$ and therefore by the choice of $q(\rho)$ we have $l_F(a_i) \le q(\rho)$, $i = -1, 0$.

Note also that
$l_F(\alpha_{\overline{e}}(c_0)) \ge 2l_X(c_0)$, $l_F(\alpha_e(c_1)) = l_X(c_1)$ and $l_X(c_0) \ge H(\rho) = 20q(\rho)$.
Thus

$$l_X(c_1) = l_F(\alpha_{\overline{e}}(c_1)) \ge$$
$$l_F(\alpha_{\overline{e}}(c_0)) - 2q(\rho) \ge 2l_X(c_0) - 2q(\rho) \ge \frac{3}{2}l_X(c_0).$$

Thus Σ is $\frac{3}{2}$-hyperbolic and the group G^* is word hyperbolic by the combination theorem.

We will now show that G is not quasiconvex in G^*. Assume that, on the contrary, G is quasiconvex in G^*. Since $F(a,b)$ is quasiconvex in G this implies that $F(a,b)$ is quasiconvex in G^*. Denote by d_* the word metric on G^* corresponding to the generating set $\mathcal{C} \cup \{t\}$. Then there is $C > 0$ such that for every $f \in F(a,b)$

$$l_F(f) \le Cl_*(f)$$

Take $f_n = t^{-n}at^n = \phi^n(a) \in F(a,b)$, $n > 0$. Then $l_F(f_n) \ge 2^n$ by the properties of ϕ and $l_*(f_n) \le 2n + l_*(a)$. Therefore

$$2^n \le l_F(f_n) \le Cl_*(f_n) \le 2C \cdot n + Cl_*(a)$$

for every integer $n > 0$. This gives us a contradiction. Therefore G is not quasiconvex in G^* and Theorem A is proved.

The remainder of this paper is devoted to proving Theorem 6.7 which was used in the proof of Theorem A and which asserts that a non-elementary torsion-free word hyperbolic group always has a free quasiconvex malnormal subgroup.

5. Malnormality in free groups.

In order to prove the existence of a free malnormal quasiconvex subgroup in a non-elementary torsion-free hyperbolic group we need to accumulate a certain amount of information about malnormal subgroups of free groups.

Our main goal in this section is to prove the following theorem (the problem of finding a malnormal subgroup in a hyperbolic group will be reduced to this statement).

Theorem D. *Let $F = F(X)$ be a nonabelian free group of finite rank with basis X. Let $H_1, \dots H_k$ be finitely generated subgroups of F of infinite index. Then there exist elements $a, b \in F$ such that*

1. *the subgroup $H = gp(a,b)$ is a free group of rank two which is malnormal in F;*
2. *no nontrivial element of H is conjugate in F to an element of $H_1 \cup \dots \cup H_k$.*

In order to prove Theorem D we will need the following series of lemmas.

Lemma 5.1. *Let $F = F(X)$ be a non-abelian finitely generated free group with a finite basis X. Let $H_1, \dots, H_k$ be finitely generated subgroups of F of infinite index.*

Then there exists an element $y \in F$ such that no nontrivial power of y is conjugate to an element of $H_1 \cup \dots \cup H_k$.

Proof.

For each subgroup H_i we choose a Nielsen reduced basis B_i (see [20] for definition and properties of Nielsen reduced sets). Let K be the maximal length of the elements of $B_1 \cup \cdots \cup B_k$. Note that K obviously has the following property. If w is a freely reduced word in X representing an element of H_i and v is a subword of w then there are words a, b in X of length at most K such that $avb \in H_i$.

We now claim that there is a nontrivial freely reduced word y in X which is not a subword of any freely reduced word w representing an element of $H_1 \cup H_2 \cup \cdots \cup H_k$.

Indeed, if there are no such y then $F = F(X)$ is the following finite union:

$$F(X) = \bigcup_{i=1}^{i=k} \bigcup_{l(a),l(b)\le K} aH_ib.$$

Note that every set aH_ib is in fact a right coset $(aH_ia^{-1})(ab)$. However all the subgroups $H_1, \dots H_k$ (and so all of their conjugates) are of infinite index in F. Thus F is covered by finitely many right cosets of subgroups of infinite index. By the result of B.H.Neumann [23], this is impossible for any group which gives us a contradiction. Thus the claim is proved and there exists a freely reduced y which is never a subword of a freely reduced word representing an element of H_i.

Moreover, we may assume that y is cyclically reduced. Suppose, on the contrary, that y is not cyclically reduced and has the form $y = x^{\epsilon}y'x^{-\epsilon}$ where $\epsilon = \pm 1$ and x is an element of X. Then we take any other letter $z \in X$, $z \neq x$ (it is possible to do since we assumed $F = F(X)$ is non-abelian and so has rank at least two). Now replace y by $y_1 = zyz$. Obviously, y_1 is cyclically reduced and y_1 is not a subword of a freely reduced word representing an element of H_i.

Thus we may indeed assume that y is cyclically reduced to begin with. We now claim that no nontrivial power of y is conjugate to an element of H_i. Indeed, suppose $q^{-1}y^pq = h \in H_i$ for some $p > 0$, $q \in F$, $1 \le i \le k$.

Take $m > 0$ to be such that $l(q) < l(y^{pm}) = pml(y)$. Then $q^{-1}y^{3pm}q = h^{3m} \in H_i$. It follows from the choice of m that the freely reduced word representing $q^{-1}y^{3pm}q = h^{3m}$ contains y as a subword which contradicts the properties of y.

Lemma 5.1 is proved.

Lemma 5.2. *Let $F = F(X)$ be a free nonabelian group of finite rank with basis X. Let a and b be nontrivial cyclically reduced words in X such that no nontrivial power of a is conjugate to a power of b. Then there exists $K > 0$ such that for any $n \neq 0, m \neq 0$ the length of the maximal subword of a^nb^m, that freely reduces to the identity, is at most K.*

Proof. Let N be the number of distinct elements of F of length at most $l(b)$. Put $K = Nl(a) = l(a^N)$. Suppose that in some product a^nb^m a terminal segment q of a^n of length at least K is freely cancelled with an initial segment of b^m. Then $q = a^{\pm N}$. This means that $a^{\pm N}$ is an initial segment of b^n. We may assume that in fact a^N is an initial segment of b^n. Put $e_0 = 1$. For every $i = 1, \dots, N$ there is an element $e_i \in F$ of length at most $l(b)$ such that a^ie_i is a power of b. All $N+1$ elements $e_0, e_1, \dots, e_N$ have length at most $l(b)$. Therefore by the choice of N there are i_1, i_2, $0 \le i_1 < i_2 \le N$ such that $e_{i_1} = e_{i_2} = e$. Thus $a^{i_1}e = b^{n_1}$ and $a^{i_2}e = b^{n_2}$ for some n_1, n_2. Therefore $e^{-1}a^{i_2-i_1}e = b^{n_2-n_1}$. However, this contradicts our assumption that no power of a is conjugate to a power of b. Thus we have proved that the length of the maximal subword of a^nb^m, that freely reduces to the identity, is at most $2K = 2Nl(a)$. Lemma 5.2 is proved.

Lemma 5.3. *Let $F = F(X)$ be a non-abelian finitely generated free group with a finite basis X. Let $H_1, \dots, H_k$ be finitely generated subgroups of F of infinite index. Let $y \in F$ be a cyclically reduced word in X such that no nontrivial power of y is conjugate to an element of $H_1 \cup \cdots \cup H_k$. Let t be another cyclically reduced word in X such that no nontrivial power of t is conjugate to a power of y.*

Then there exists integer $m > 0$ with the following property. If $n \ge m$ then no nontrivial element of the subgroup $D_n = gp(y^nt^{3n}y^n, t^ny^{3n}t^n)$ is conjugate to an element of $H_1 \cup \cdots \cup H_k$.

Proof of Lemma 5.3.

Let $K_1 > l(y) + l(t)$ be such that for any $k \neq 0, s \neq 0$ the length of the maximal subword of y^kt^s that freely reduces to the identity is at most K_1. The existence of such a constant K_1 follows from Lemma 5.2. For each subgroup H_i we choose a Nielsen reduced basis B_i. Let K_2 be the maximal length of the elements of $B_1 \cup \dots \cup B_k$. Put $K = max(K_1, K_2)$. Note that by the choice of y, t the subgroup $D = gp(y, t) \leq F$ is free of rank two and therefore for every $n > 0$ the group $D_n = gp(y^n t^{3n} y^n, t^n y^{3n} t^n)$ is also a free group of rank two with basis $y^n t^{3n} y^n, t^n y^{3n} t^n$.

Let $m_1 > 0$ be such that $l(y^{m_1}) = m_1 l(y) > K \geq K_1$ and $l(t^{m_1}) = m_1 l(t) > K \geq K_1$. Then it is clear that for every $n \geq 3m_1$ every freely reduced word w representing a nontrivial element of D_n contains y^{n-2m_1} as a subword. Moreover, for every $n \geq 3m_1$, any $z \in F$ and any nontrivial element $d \in D_n$ some sufficiently high power of $z^{-1}dz$, when written in the freely reduced form, contains y^{n-2m_1} as a subword.

Let N be the number of distinct elements of F of length at most K. Put

$$m = 3m_1 + N + 1.$$

Suppose now that $n \geq m$ and $d \in D_n, d \neq 1$.

We claim that no nontrivial power of d is conjugate in F to an element of H_i. Indeed, suppose $z^{-1}d^p z = h \in H_i$ for some $z \in F$, $p > 0$. Then by the remarks above some sufficiently high power h^s of h contains, when written in a freely reduced form, a subword y^{n-2m_1}.

Note also that by the properties of a Nielsen basis for H_i, if v is an initial segment of a freely reduced word in X representing an element of H_i, then for some element $a \in F$ with $l(a) \leq K_2 \leq K$ we have $va \in H_i$.

Let the freely reduced word w representing h^s have the form $w \equiv vy^{n-2m_1}v'$. Note that $n \geq m = 3m_1 + N + 1$ and therefore $n - 2m_1 \geq N + 1$. Thus there are some elements $a_0, a_1, \dots a_N \in F$ with $l(a_j) \leq K_2 \leq K$ such that $vy^j a_j \in H_i$ for every $j = 0, 1, \dots N$. Since there are only N different elements of length at most K in F, there exist $j_1 < j_2$, $0 \leq j_1, j_2 \leq N$ such that $a_{j_1} = a_{j_2} = a$. Thus we have $vy^{j_1}a \in H_i$ and $vy^{j_2}a \in H_i$ which implies $a^{-1}y^{j_2-j_1}a \in H_i$. This contradicts our choice of y.

We have shown that if $n \geq m = 3m_1 + N + 1$ then no nontrivial element of D_n is conjugate to an element of $H_1 \cup \dots \cup H_k$. Lemma 5.3 is proved.

Lemma 5.4. *Let $F(y, t)$ be the free group on y, t. Then the subgroup*

$$M = gp(yt^3y, ty^3t)$$

is malnormal in $F(y, t)$.

Proof. This fact can be established by elementary means, however the proof is a rather lengthy exercise. We have verified the correctness of Lemma 5.4 using the computational group theory software package MAGNUS [21].

Lemma 5.5. *Let $F = F(a, b)$ be the free group on a, b and let $n > 0$, $m > 0$ be some integers. Let $L = gp(a^n, b^m)$ be the subgroup of F generated by a^n, b^m. Suppose that $zhz^{-1} = h'$ for some $h, h' \in L$, $h \neq 1$, $z \in F - L$. Then h is conjugate in L either to a power of a^n or to a power of b^m.*

Proof. First notice that it is enough to prove Lemma 5.5 for those z which are shortest in their coset classes zL. Indeed, suppose that Lemma 5.5 has been established for such z. Now let z be an arbitrary element as in Lemma 5.5. Then for some $h_0 \in L$ we have $z = z_1 h_0$ where z_1 is shortest in the coset class zL. Then $zhz^{-1} = h'$ implies that $z_1(h_0 h h_0^{-1})z_1^{-1} = h'$. Therefore by our assumption $h_0 h h_0^{-1}$ is conjugate in L to a power of a^n or a power of b^n and the same is true for h.

Thus from now on we will assume that z is shortest in zL. Without a loss of generality we may assume that the last letter of z (when written as a freely reduced word in a, b) is $a^{\pm 1}$. If z is a power of a then the statement of Lemma 5.5 is obvious. If z is not a power of a then $z \equiv qb^k a^s$ where $k \neq 0$, $s \neq 0$. Moreover, since z is shortest in zL, it cannot be shortened by multiplying on the right by $a^{\pm n}$. Therefore $0 < |s| < n$.

If h is a power of a^n or a power of b^m, then there is nothing to prove. Suppose therefore that h is not a power of a^n and that h is not a power of b^m. Then $h = x_1 x_2 \dots x_p$, where $p \geq 2$, each of x_i is a nonzero power of a^n or b^m and the sequence strictly alternates.

There are several cases to consider.

Case 1 Both x_1 and x_p are powers of b^m. Then $zhz^{-1} \equiv qb^k a^s x_1 x_2 \dots x_p a^{-s} b^{-k} q^{-1}$ is freely reduced as written. Since $0 < |s| < n$, this implies $zhz^{-1} \notin L = gp(a^n, b^m)$ which contradicts our assumptions.

Case 2 Both x_1 and x_p are powers of a. Then $p \geq 3$ and $x_1 = a^{n_1 n}$, $x_2 = a^{n_2 n}$. In this case the freely reduced form of zhz^{-1} is $qb^k a^{n_1 n+s} x_2 \dots x_{p-1} a^{n_2 n-s} b^{-k} q^{-1}$. Again, since $0 < |s| < n$, both numbers $n_1 n+s$ and $n_2 n - s$ are not divisible by n. Therefore $zhz^{-1} \notin L = gp(a^n, b^m)$ which contradicts our assumptions.

Case 3 The word x_1 is a power of a^n and the word x_p is a power of b^m. Then $x_1 = a^{n_1 n}$. In this case the freely reduced form of zhz^{-1} is $qb^k a^{n_1 n+s} \dots x_p a^{-s} b^{-k} q^{-1}$. Once again, since $0 < |s| < n$, we conclude that both s and $n_1 n + s$ are not divisible by n. Thus $zhz^{-1} \notin L = gp(a^n, b^m)$ which contradicts our assumptions.

Case 4 The word x_1 is a power of b^m and the word x_2 is a power of a^n. This case reduces to Case 3 if we replace the equality $zhz^{-1} = h'$ by $zh^{-1}z^{-1} = (h')^{-1}$.

Thus we proved that $p > 1$ is impossible, that is h is a power of a^n or a power of b^m. This completes the proof of Lemma 5.5.

Lemma 5.6. *Let $F = F(a, b)$ be the free group on a, b. Then for any $n > 0, m > 0$ the subgroup $M = gp(a^n b^{3m} a^n, b^m a^{3n} b^m)$ is malnormal in F.*

Proof. Let $L = gp(a^n, b^m)$. Observe first that if $z^{-1}hz = h'$ where $h, h' \in L - 1$, $z \in F - L$ then by Lemma 5.5 h is conjugate to either a power of a or a power of b.

Suppose now that M is not malnormal in F. Then for some nontrivial $h \in M$ and some $z \in F - M$ we have $z^{-1}hz \in M$. By Lemma 5.4 the subgroup M is malnormal in L and therefore $z \notin L$. Then by the previous observation h is conjugate to either a power of a or a power of b. Clearly, neither of these cases are possible which gives us a contradiction.

Thus M is malnormal in F which completes the proof of Lemma 5.6.

Lemma 5.7. *Let $F = F(X)$ be a free group on X. Let v and s be nontrivial freely reduced words in X such that $l(v) > l(s)$.*

Suppose also that $sv \equiv v\alpha$, where $l(\alpha) = l(s)$ and the products sv and $v\alpha$ are freely reduced as written. Let n be the biggest integer such that $nl(s) \leq v$. Then $v \equiv s^n q$ where the product $s^n q = sss \dots sq$ is freely reduced as written and $l(q) < l(s)$.

Proof. We will prove Lemma 5.7 by induction on n. When $n = 1$, the statement is obvious. Suppose now that $n > 1$ and that the statement has been proved for all smaller values of n.

Note that since $sv \equiv v\alpha$ and $l(v) > l(s)$, the word s is a proper initial segment of v. Thus v has the form $v \equiv sv_1$. Therefore $sv \equiv v\alpha$ implies $ssv_1 \equiv sv_1\alpha$. Hence s is cyclically reduced and $sv_1 \equiv v_1\alpha$. Note that since $n > 1$, $l(v) \geq 2l(s)$ and so $l(v_1) \geq l(s)$.

If $l(v_1) = l(s)$ then $v_1 = s$, $v = s^2$ and the statement of Lemma 5.7 obviously holds.

If $l(v_1) > l(s)$ then by the inductive hypothesis $v_1 \equiv s^{n-1}q$ with $l(q) < l(s)$. Therefore $v \equiv sv_1 \equiv ss^{n-1}q \equiv s^n q$. Once again the statement of Lemma 5.7 holds.

Lemma 5.8. *Let $F = F(X)$ be a free non-Abelian group of finite rank with basis X. Let a, b be cyclically reduced words in X such that no nontrivial power of a is conjugate in F to a power of b. Then there exists some number $M > 0$ such that a^M is not a subword of a power of b and b^M is not a subword of a power of a.*

Proof.

Let N be the number of elements of F of length at most $l(b)$. Suppose that a^N is a subword of a power of b, say b^n. That is, va^N is an initial segment of b^n for some v. For each $i = 0, \dots, N$ there exists an element $e_i \in F$ with $l(e_i) \leq l(b)$ such that $va^i e_i$ is a power of b, that is $va^i e_i = b^{j_i}$. All $N+1$ elements $e_0, \dots e_N$ have length at most $l(b)$. Therefore by the choice of N there are i_1, i_2, $0 \leq i_1 < i_2 \leq N$ such that $e_{i_1} = e_{i_2} = e$. Then $va^{i_1}e = b^k$, $va^{i_2}e = b^s$ for some k, s. Therefore $e^{-1}a^{i_2 - i_1}e = b^{s-k}$ with $i_2 - i_1 > 0$. This contradicts our assumption that no power of a is conjugate to a power of b.

Thus a^N is never a subword of a power of b. By symmetry there is $N_1 > 0$ such that b^{N_1} is never a subword of a power of a. Then $M = \max\{N, N_1\}$ satisfies the requirements of Lemma 5.8.

Lemma 5.9. *Let $F = F(X)$ be a free non-Abelian group with basis X. Let a be a nontrivial cyclically reduced word in X. Then a is not a subword of a^{-n} for $n > 0$ and a^{-1} is not a subword of a^n for $n > 0$.*

Proof. Suppose that a is a subword of a^{-n} where $n > 0$. Since $l(a) = l(a^{-1})$ this implies that a is in fact a cyclic permutation of a^{-1}, that is $z^{-1}az = a^{-1}$ for some $z \in F$. Clearly z does not commute with a since a has infinite order. Therefore the subgroup $H = gp(a, z)$ of F is free of rank 2 with basis a, z. However in the free group on a, z we obviously have $z^{-1}az \neq a^{-1}$ which gives us a contradiction. Lemma 5.9 is proved.

Lemma 5.10. *Let $F = F(X)$ be a free non-Abelian group of finite rank with basis X. Let a and s be nontrivial cyclically reduced words in X. Let N be the number of distinct elements of F of length at most $l(a)$. Suppose that s^N is a subword of a^n for some n. Then a nontrivial power of s is conjugate to a power of a.*

Proof. The proof is exactly the same as the proof of Lemma 5.8 given above.

Lemma 5.11. *Let $F = F(X)$ be a free group of finite rank and let a be a nontrivial root-free cyclically reduced word in X. Let N be the number of elements in $F(X)$ of length at most $l(a)$. Suppose s is a nontrivial cyclically reduced word such that some positive power a^n of a has initial segment s^{N+1}. Then $s = a^j$ for some $j > 0$.*

Proof. Suppose s and a are as above and so s^{N+1} is an initial segment of a^n. Then for every $i = 1, 2, \dots N$ there is a word e_i with $l(e_i) \leq l(a)$ such that $s^i e_i = a^{n_i}$ for some $0 \leq n_i \leq n$. by the choice of N this implies that there are some i_1, i_2, $1 \leq i_1 < i_2 \leq N$ such that $e_{i_1} = e_{i_2} = e$. Then $e = s^{-i_1}a^{n_{i_1}}$ and $ea^{n_{i_2}-n_{i_1}}e^{-1} = s^{i_2-i_1}$. Hence

$$s^{-i_1}a^{n_{i_1}}a^{n_{i_2}-n_{i_1}}a^{-n_{i_1}}s^{i_1} = s^{i_2-i_1}$$

and

$$a^{n_{i_2}-n_{i_1}} = s^{i_2-i_1}.$$

Since a is a root-free element and the subgroup $gp(a)$ is malnormal in F, this implies that $s = a^j$ for some $j > 0$. Lemma 5.11 is proved.

Lemma 5.12. *Let $F = F(X)$ be a free group of finite rank and let a be a nontrivial root-free cyclically reduced word in X. Let N be the number of elements in $F(X)$ of length at most $l(a)$. Suppose s is a nontrivial freely reduced word in X such that the product $s \cdot a$ is freely reduced and such that sa^n has initial segment a^n for some n with $nl(a) \geq (N+1)l(s)$. Then $s \equiv a^j$ for some $j > 0$.*

Proof. Lemma 5.7 implies that s is cyclically reduced and that $a^n \equiv s^k\alpha$ with $l(\alpha) < l(s)$. Thus

$$(k+1)l(s) > l(a^n) = nl(a) \geq (N+1)l(s) \text{ by the choice of } n$$

and therefore $k \geq N+1$. Since a^n has initial segment s^k, Lemma 5.11 implies that $s = a^j$ for some $j > 0$. Lemma 5.12 is proved.

Proposition 5.13. *Let $F = F(X)$ be a non-abelian free group of finite rank and let $a \in F$ be a non-trivial cyclically reduced root-free word in X. Then there exists a positive integer K such that the following holds.*

Let $n \geq K$ and let $H = < a^n > = gp(a^n)$. Let $g \in F$ be a shortest element in the double coset HgH and such that the element $g \cdot a^n$, when freely reduced over X, has $a^{[5n/8]}$ as its initial segment.

Then $g \in gp(a) = < a >$.

Proof. Let N be the number of elements of $F(X)$ of length at most $l(a)$. Put $K = 100(N+1)$ and assume that $n \geq K$.

Let $g \equiv xy^{-1}$ and $a^n \equiv yz$ where y is the maximal initial segment of a^n that is freely cancelled in $g \cdot a^n$. Note that $l(y) \leq (1/2)l(a^n)$ because g is shortest in gH.

We have that $ga^n \equiv xz$ has $a^{[5n/8]}$ as its initial segment. Thus x has length $l(x) \leq (1/2)l(a^n)$ because $g \equiv xy^{-1}$ is shortest in Hg. This implies, in particular, that x is an initial segment of $a^{[5n/8]}$. We also conclude that $l(g) \leq l(x) + l(y) \leq l(a^n)$.

Since y is an initial segment of a^n with $l(y) \le l(a^n)/2$, it has the form

$$y \equiv a^k a', \quad \text{where} \quad a^n \equiv a^k a' a'' a^{n-k-1}, \quad a \equiv a'a'', \quad \text{and } k \le n/2.$$

Similarly, since x is an initial segment of $a^{[5n/8]}$ of length at most $l(a^n)/2$, we have

$$x \equiv a^s a_1, \quad \text{where} \quad a^n \equiv a^s a_1 a_2 a^{n-s-1}, \quad a \equiv a_1 a_2, \quad \text{and } s \le n/2.$$

Thus $g \equiv xy^{-1} \equiv a^s a_1 (a')^{-1} a^{-k}$ and

$$ga^n = a^s a_1 (a')^{-1} a^{-k} \cdot a^k a' a'' a^{n-k-1} \equiv a^s a_1 a'' a^{n-k-1}.$$

Since ga^n starts with $a^{[5n/8]}$ and $s \le n/2$ this implies that $a_1 a'' a^{n-k-1}$ has initial segment $a^{[n/8]}$. Since $k \le n/2$ this implies that $a_1 a'' a^{[n/8]}$ has initial segment $a^{[n/8]}$. Note that a_1 and a'' are segments of a, so that $l(a_1 a'') \le 2l(a)$. By the choice of $n \ge K = 100(N+1)$ we have

$$[n/8] l(a) > (N+1)(2l(a)) \ge (N+1) l(a_1 a'').$$

Therefore Lemma 5.12 implies that $a_1 a'' \equiv a^j$ for some $0 \le j \le 2$.

Case 1. Suppose that $j = 2$. Then $l(a_1) = l(a'') = l(a)$ and therefore $a_1 = a'' = a$, $a_2 = a' = 1$. Hence $g = a^s a_1 (a')^{-1} a^{-k} = a^s a a^{-k} = a^{s-k+1}$ and $g \in gp(a)$ as required.

Case 2. Suppose that $j = 1$ and $a_1 a'' \equiv a$. Since $a_1 a_2 \equiv a' a'' \equiv a$, this implies that $a' = a_1, a'' = a_2$. Hence $g = a^s a_1 (a')^{-1} a^{-k} = a^s a_1 (a_1)^{-1} a^{-k} = a^{s-k} \in gp(a)$ as required.

Case 3. Suppose that $j = 0$ and $a_1 = a'' = 1$. Then $a_2 = a' = a$ and hence $g = a^s a_1 (a')^{-1} a^{-k} = a^s a^{-1} a^{-k} = a^{s-k-1} \in gp(a)$ as required.

This completes the proof of Proposition 5.13.

Proposition 5.14. *Let $F = F(X)$ be a non-abelian free group of finite rank. Let a and b be nontrivial cyclically reduced elements of F such that no nontrivial power of a is conjugate in F to a power of b. Then there exists an integer $N > 1$ such that for any $n \ge N, m \ge N$, $A = a^n$, $B = B^m$, $H = gp(a^n, b^m) = gp(A, B)$ the following holds.*

If $g \in F$ is shortest in the double coset HgH and if $gHg^{-1} \cap H \ne 1$ then $g \in gp(a) \cup gp(b)$.

Proof. Let K_0 be such that

1. the length of the maximal initial segment of b^j which may be freely cancelled in $a^i \cdot b^j$, $i, j \in \mathbb{Z}$ is less than K_0;
2. $K_0 l(a) \ge l(b)$ and $K_0 l(b) \ge l(a)$
3. a^{K_0} is not a subword of a power of b and b^{K_0} is not a subword of a power of a.

Let K_a and K_b be the constants provided by Proposition 1 for a and b respectively. Then we put $N = 1000 K_0 \max\{K_a, K_b\} l(a) l(b)$. Let $n, m \ge N$, $A = a^n$, $B = b^m$, $H = gp(A, B)$. Let g be as in Proposition 5.14.

Notice that the choice of $n, m \ge N$ implies that the length of the maximal initial segment of $B^{\pm 1}$ that may be freely cancelled in the various products $A^{\pm 1} B^{\pm 1}$ is less than $(1/1000) \min\{l(A), l(B)\}$.

Since $gHg^{-1} \cap H \ne 1$ we have that $gW(A,B)g^{-1} = W_1(A,B)$ for some freely reduced words W, W_1 in A, B. By taking powers, if necessary, we may assume that the length of W (as an A,B-word) is greater than $1000 l(g)$. We will rewrite the equality above as

$$gW(A,B) = W_1(A,B) g = f.$$

Without loss of generality we may assume that W starts with A. There are several cases to consider.

Case 1. The word $W_1(A,B)$ starts with A. Then f has initial segment $a^{[999/1000n]}$. Let $g \equiv xy^{-1}$ where y is the maximal initial segment of A freely cancelled in $g \cdot A$. Since g is shortest in $gp(A,B) \cdot g \cdot gp(A,B)$, we have $l(g) \le (n/2) l(a) = 1/2 l(A)$.

Subcase 1.a. Suppose that $l(x) > 251/1000nl(a)$. Then $A = a^n = yzq$ where q is the terminal segment of A that is cancelled in the product of A with the second letter of W. As we noted before, we have $l(q) < (1/1000)l(A)$ and $l(y) \le (1/2)l(A)$. Thus $l(z) > 499/1000l(A)$, $l(x) > 251/1000l(A)$ and so $l(xz) > 750/1000l(A) = 3/4l(A)$.

Obviously, xz is an initial segment of $gW(A,B) = f$ and of gA of length greater than $[3/4n]l(a)$. We have also seen that $gW(A,B) = f$ has initial segment $a^{[999/1000n]}$. Thus $gA = ga^n$ has initial segment $a^{[3/4n]}$. Since g is shortest in $gp(A,B) \cdot g \cdot gp(A,B)$, the element g is also shortest in $gp(A) \cdot g \cdot gp(A)$. Therefore by Proposition 5.13 we have $g \in gp(a) \subset gp(a) \cup gp(b)$ as required.

Subcase 1.b. Suppose that $l(x) \le 251/1000l(A)$.

If $l(y) > 251/1000l(A)$ then we may replace the identity $gW(A,B) = W_1(A,B)g$ by $g^{-1}W_1(A,B) = W(A,B)g^{-1}$. Since g^{-1} is obviously also shortest in $Hg^{-1}H$, we observe that this case is symmetric to Subcase 1.a and so $g^{-1} \in gp(a), g \in gp(a)$.

Suppose, on the contrary, that $l(y) \le 251/1000l(A)$. Recall that $A = yzq$ where $l(q) < 1/1000l(A)$. Therefore $l(z) > 748/1000l(A)$ and so $l(xz) > 748/100l(A)$. We see that both $f = gW = W_1g$ and gA have initial segment xz of length greater than $748/1000l(A)$. On the other hand $gW = f$ has initial segment $a^{[999/1000n]}$. Therefore $gA = ga^n$ has initial segment $a^{[748/1000n]}$ and so $a^{[5/8n]}$. Since g is shortest in $gp(A,B) \cdot g \cdot gp(A,B)$, the element g is also shortest in $gp(A) \cdot g \cdot gp(A)$. Therefore by Proposition 5.13 we have $g \in gp(a) \subset gp(a) \cup gp(b)$ as required.

Case 2. Suppose that the first letter of $W_1(A,B)$ is B.

Recall that $g = xy^{-1}$, $A = yzq$ where

$$l(q) < (1/1000)l(A), l(y) \le 1/2l(A), l(z) > (499/1000)l(A).$$

Also $gW = f$ has initial segment xz. On the other hand $f = W_1g$ has initial segment $b^{[999/1000n]}$ because $W_1(A,B)$ starts with B. Hence $l(x) \le 1/2l(B)$ because otherwise $g \equiv xy^{-1}$ would not have been shortest in $gp(B) \cdot g \cdot gp(B)$. Thus x is an initial segment of B with $l(x) \le 1/2l(B)$. Since $l(z) > 499/1000l(A)$, the choice of $n \ge N$ implies that $l(z) > l(b)(N_0 + 1)$. Let z' be the initial segment of z of length $l(b)(N_0 + 1)$.

Then xz' and $b^{[999/1000m]}$ are initial segment of f where $l(x) \le 1/2ml(b)$. Therefore z' is a subword of $b^{[999/1000m]}$ and of a^n of length $l(b)(N_0 + 1)$. This implies that b^{N_0} is a subword of a^n which is impossible by the choice of N_0.

Case 3. Suppose the first letter of $W_1(A,B)$ is B^{-1}. This case is completely analogous to Case 2.

Case 4. Suppose the first letter of $W_1(A,B)$ is A^{-1}. Then, once again, $g \equiv xy^{-1}$, $A = yzq$ where

$$l(q) < 1/1000l(A), l(y) \le 1/2l(A), l(z) > 499/1000l(A).$$

Also $gW = f$ has initial segment xz. On the other hand $f = W_1g$ starts with $a^{-[999/1000n]}$. Hence $l(x) \le 1/2l(A^{-1}) = l(A)$ since $g = xy^{-1}$ is shortest in $gp(A) \cdot g \cdot gp(A)$. Let $\hat{z}$ be an initial segment of z of length $2l(a)$ (and so $\hat{z}$ contains the subword a). Then $\hat{z}$ is a subword of $a^{-[999/1000n]}$ and so a is a subword of a^{-n}, which is impossible by Lemma 5.9.

This completes the proof of Proposition 5.14.

Proposition 5.15. *Let $F = F(X)$ be a non-abelian free group of finite rank. Let $a, b \in F$ be non-trivial root-free elements such that no non-trivial power of a is conjugate to a power of b. Then there exists an integer $N > 1$ with the following properties.*

Suppose $m, n \ge N$ and let $D = gp(a^n b^{3m} a^n, b^m a^{3n} b^m)$. Then D is malnormal in F.

Proof. Let N be the constant provided by Proposition 5.14. Suppose that D is not malnormal in F and there is $g \in F - D$ such that $gDg^{-1} \cap D \ne 1$. Put $H = gp(a^n, b^m)$, so that $D \le H$. Let g' be the shortest element in HgH so that $g = h_1g'h_2$ for some $h_1, h_2 \in H$. The fact that $gDg^{-1} \cap D \ne 1$ implies $gHg^{-1} \cap H \ne 1$. Thus

$$h_1g'h_2Hh_2^{-1}(g')^{-1}h_1^{-1} \cap H \ne 1$$
$$h_1g'H(g')^{-1}h_1^{-1} \cap H \ne 1 \quad \text{and so}$$
$$g'H(g')^{-1} \cap H \ne 1$$

By Proposition 5.14 this means that $g' \in gp(a) \cup gp(b)$ and so $g' \in gp(a,b) = L$. Since $H = gp(a^n, b^m) \leq gp(a,b) = L$ and $g = h_1 g' h_2$ with $h_1, h_2 \in H$, we have $g \in L$. Thus $D \leq L$, $gDg^{-1} \cap D \neq 1$ and $g \in L$. However, D is malnormal in L by Lemma 5.6. Therefore $g \in D$ which contradicts our assumption that $g \in F - D$. Proposition 5.15 is proved.

Theorem 5.16 (Theorem D). *Let $F = F(X)$ be a non-Abelian free group of finite rank with basis X. Let $H_1, \dots H_k$ be finitely generated subgroups of F of infinite index. Then there exist elements $a, b \in F$ such that*

1. *the subgroup $H = gp(a,b)$ is a free group of rank two which is malnormal in F;*
2. *no nontrivial element of H is conjugate in F to an element of $H_1 \cup \dots \cup H_k$.*

Proof. By Lemma 5.1 there exists $y \in F$ such that no nontrivial power of y is conjugate in F to an element of $H_1 \cup \dots \cup H_k$. Take t to be any cyclically reduced word in X such that no power of t is conjugate in F to a power of y. (Since $|X| > 2$ we may even choose $y \in X$.) Put $D_n = gp(y^n t^{3n} y^n, t^n y^{3n} t^n) \leq F$ where $n > 0$. By Lemma 5.3 there exists $M_1 > 0$ such that if $n \geq M_1$ then no nontrivial element of D_n is conjugate in F to an element of $H_1 \cup \dots \cup H_k$. By Proposition 5.15 there is $M_2 > 0$ such that in $n \geq M_2$ then D_n is malnormal in F. Put $M = \max\{M_1, M_2\}$. Then for any $n \geq M$ the subgroup D_n of F satisfies the requirements of Theorem 5.16.

6. Quasiconvex subgroups and malnormality

In this section we will show that if G is a torsion-free hyperbolic group and H is a non-elementary subgroup of G then H contains a free subgroup F of rank two such that F is malnormal and quasiconvex in G.

First, we need to accumulate a certain amount of information about the properties of quasiconvex subgroups.

Lemma 6.1. *Let G be a word hyperbolic group with a finite generating set X and word metric d_X. Let H be a quasiconvex subgroup of G. Then there exists $\lambda > 0$ with the following properties.*

Suppose that $g \in G$ is such that $g \notin H$ and that g is shortest with respect to d_X in the double coset class HgH. Choose w to be an X-geodesic representative of g. Then for any X-geodesic words v_1 and v_2 representing elements of H the word $v_1 w v_2$ is (λ, λ)-quasigeodesic with respect to d_X (see [6] for definition of quasigeodesics).

Proof.

The statement of Lemma 6.1 easily follows from the proof of Lemma 4.5 in [6] and we omit the details.

Remark. Suppose that G is a group and M is a subgroup of G. Let $z \in G$ and $z' = m_1 z m_2$ for some $m_1, m_2 \in M$. Then $zMz^{-1} \cap M \neq 1$ if and only if $z'Mz'^{-1} \cap M \neq 1$.

The following lemma shows that if H is a quasiconvex subgroup of a torsion-free hyperbolic group G then there are at most finitely many double cosets HzH that "violate" malnormality of H.

Lemma 6.2. *Let G be a torsion-free word hyperbolic group with a finite generating set X and word metric d_X. Let H be a nontrivial quasiconvex subgroup of G. Then there are only finitely many double cosets HzH of elements $z \in G$, $z \notin H$ such that $z^{-1}Hz \cap H \neq 1$.*

Proof. Let $\mathcal{H} = \{h_1, \dots h_k\}$ be a finite generating set of H. For every $i = 1, \dots k$ pick a d_X-geodesic word v_i representing h_i. Since H is quasiconvex in G, there is $\lambda_1 > 0$ such that if $W(h_1, \dots, h_k)$ is an $\mathcal{H}$-geodesic word then $W(v_1, \dots, v_k)$ is a (λ_1, λ_1)-quasigeodesic with respect to d_X.

Recall also that by Lemma 6.1 there is $\lambda > 0$ such that if v is an X-geodesic representative of an element $z \in G$, $z \notin L$ which is d_X-shortest in HzH, then for any X-geodesic words q_1, q_2 representing elements of H the word $q_1 v q_2$ is (λ, λ)-quasigeodesic with respect to d_X.

Put $\lambda' = \max\{\lambda, \lambda_1\}$. Let $\epsilon > 1$ be such that any two (λ', λ')-quasigeodesic paths in the Cayley graph $\Gamma(G, X)$ with common endpoints are ϵ-Hausdorff close. Let K be the maximal length of the words $v_1, \dots v_k$.

Suppose now that there are infinitely many double cosets HzH such that

$$zHz^{-1} \cap H \neq 1.$$

Then there exists $z \in G$, $z \notin H$ such that $zHz^{-1} \cap H \neq 1$, z is shortest in HzH and $l_X(z) > 2\epsilon + K + 2$. Let Z be a d_X-geodesic representative of z.

Then $zhz^{-1} = h'$ and $zh = h'z$ for some $h, h' \in H$. Replacing h and h' by some high powers of themselves we can assume that $l_X(h') > 3\epsilon + K + 3$.

Choose a d_X-geodesic representative V for h and a d_X-geodesic representative V' for h'. Then both words ZV and $V'Z$ are (λ, λ)-quasigeodesic with respect to d_X representing the element $zh = h'z$. Recall that $l_X(z) > 2\epsilon + K + 2$. Take Z_1 to be the initial segment of Z of length $2\epsilon + K + 2$ so that $Z \equiv Z_1 Z_2$ (see Figure 6.1).

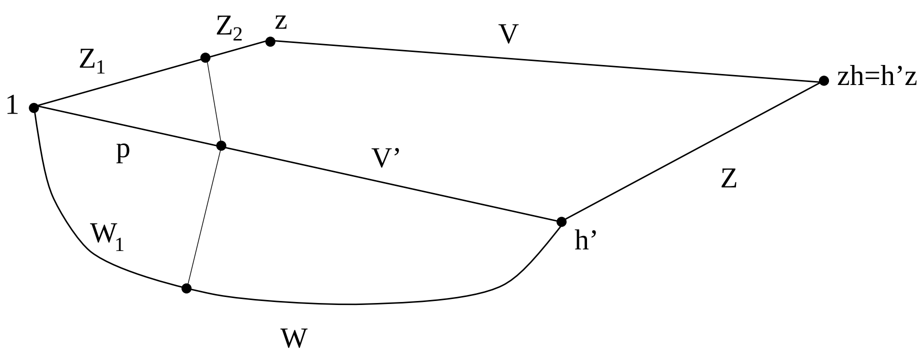


Figure 6.1

By the choice of ϵ there is an initial segment p of the word $V'Z$ such that $d_X(\overline{Z_1}, \overline{p}) \leq \epsilon + 1$. Since $l(Z_1) = 2\epsilon + K + 2$, this implies that $l_X(\overline{p}) \leq 2\epsilon + K + 2 + \epsilon + 1 = 3\epsilon + K + 3$. Note that h' was chosen so that $l_X(h') > 3\epsilon + K + 3$. Therefore the initial segment p of $V'Z$ is in fact an initial segment of V'.

Note that $zh = h'z$. Choose a $d_{\mathcal{H}}$ geodesic representative $W(h_1, \dots h_k)$ of h'. Denote the X-word $W(v_1, \dots, v_k)$ by U. Then both U and V' represent the element h' of G. Moreover, both words U and V' are (λ', λ')-quasigeodesic with respect to d_X and so are ϵ-Hausdorff close in $\Gamma(G, X)$. Therefore by the choice of ϵ and K there is an initial segment $W_1(h_1, \dots, h_k)$ of $W(h_1, \dots, h_k)$ such that $d_X(\overline{p}, \overline{W_1}) \leq \epsilon + K$. Therefore $d_X(\overline{Z_1}, \overline{W_1}) \leq d_X(\overline{Z_1}, \overline{p}) + d_X(\overline{p}, \overline{W_1}) \leq \epsilon + 1 + \epsilon + K = 2\epsilon + K + 1$.

Denote $q = (\overline{W_1})^{-1}\overline{Z_1} \in G$. Then

$$l_X(q\overline{Z_2}) \leq 2\epsilon + K + 1 + l(Z_2) < 2\epsilon + K + 2 + l(Z_2) = l(Z_1) + l(Z_2) = l(Z) = l_X(z).$$

On the other hand $\overline{W_1} \cdot q\overline{Z_2} = z$ and $\overline{W_1} \in H$. Thus $Hz = Hq\overline{Z_2}$ and so $HzH = Hq\overline{Z_2}H$. But we have established that $l_X(q\overline{Z_2}) < l_X(z)$. This gives us a contradiction with the choice of z as a shortest element in HzH. Lemma 6.2 is proved.

Lemma 6.3. *Let G be a torsion-free word hyperbolic group and let H be a nontrivial quasiconvex subgroup of G. Let $g \in G$, $g \notin H$ be such that $g^i H g^{-i} \cap H \neq 1$ for every $i > 0$. Then for some $n > 0$ we have $g^n \in H$.*

Proof. This statement is an immediate corollary of the main result of the paper [15] by R.Gitik, M.Mitra, M.Sageev and E.Rips.

The following statement is essentially due to M.Gromov [13, Theorem 5.3.E]. For a careful argument the reader is referred to Lemma 1.1 and Lemma 1.2 in the paper of T.Delzant [9].

Lemma 6.4. *Let G be a word hyperbolic group. Let $a, b \in G$ be such that no nontrivial power of a is equal to a power of b. Then there is $n > 0$ such that the subgroup $H = gp(a^n, b^n)$ is free of rank two and is quasiconvex in G.*

Lemma 6.5. *Let G be a torsion-free word hyperbolic group. Let $H = F(a,b)$ be a quasiconvex subgroup of G which is free group with basis a,b. Put $V_G(H)$ to be the virtual normalizer of H in G, that is*

$$V_G(H) = \{g \in G \mid \ |H : H \cap gHg^{-1}| < \infty, |gHg^{-1} : H \cap gHg^{-1}| < \infty\}.$$

Then $H = V_G(H)$.

Proof. Since H is infinite and quasiconvex in G, the subgroup H has finite index in its virtual normalizer $V_G(H)$ (see [22], [19, Theorem 1(3)]]. Also, G is torsion-free and H is a free group. Thus $V_G(H)$ is a torsion-free group that has a free subgroup of finite index. By the theorem of J.Stallings [31] this implies that $V_G(H)$ is itself a free group. Suppose that $V_G(H) \neq H$, that is the index of H in $V_G(H)$ is greater than 1. By the Theorem II of [17, ch. 36] this implies that the rank of $V_G(H)$ is strictly less than the rank of H. However the rank of H is equal to two and $H \leq V_G(H)$ which gives us a contradiction. Thus $H = V_G(H)$ and Lemma 6.5 is proved.

Lemma 6.6. *Let G be a group and let H be a subgroup of G. Let $g \in G$ and suppose that the subgroup $K_1 = gHg^{-1} \cap H$ has finite index in H and infinite index in gHg^{-1}. Then for any integer $n > 0$ the subgroup $K_n = g^nHg^{-n} \cap H$ has finite index in H and infinite index in g^nHg^{-n}.*

Proof. We will prove the statement by induction on n. When $n = 1$, the statement is obvious. Suppose now $n > 1$ and that Lemma 6.6 has been established for all smaller values.

Thus by the inductive hypothesis the subgroup $K_{n-1} = g^{n-1}Hg^{-(n-1)} \cap H$ has finite index in H and infinite index in $g^{n-1}Hg^{-(n-1)}$. Conjugating everything by g we conclude that $gK_{n-1}g^{-1} = g^nHg^{-n} \cap gHg^{-1}$ has finite index in gHg^{-1} and infinite index in g^nHg^{-n}.

We know that the subgroup $K_1 = gHg^{-1} \cap H$ has finite index in H. Also the intersection of gHg^{-1} and g^nHg^{-n} has infinite index in g^nHg^{-n}. Therefore the intersection of any subgroup of gHg^{-1} and g^nHg^{-n} has infinite index in g^nHg^{-n}. In particular $K_1 \cap g^nHg^{-n}$ has infinite index in g^nHg^{-n}. Since K_1 is a subgroup of finite index in H, this implies that $K_n = H \cap g^nHg^{-n}$ has infinite index in g^nHg^{-n}.

On the other hand we already observed that $L_1 = gK_{n-1}g^{-1} = g^nHg^{-n} \cap gHg^{-1}$ has finite index in gHg^{-1}. Recall that K_1 is a subgroup of gHg^{-1} and therefore $K_1 \cap L_1$ has finite index in K_1. Since $K_1 = gHg^{-1} \cap H$ is of finite index in H, this implies that $K_1 \cap L_1$ is of finite index in H. However $K_1 \cap L_1 \leq L_1 \leq g^nHg^{-n}$ and therefore $g^nHg^{-n} \cap H$ is of finite index in H. This completes the inductive step and the proof of Lemma 6.6.

Theorem 6.7. *Let G be a torsion-free word hyperbolic group. Let Γ be a non-elementary subgroup of G. Then there is a subgroup $F \leq \Gamma$ such that F is a free group of rank 2 , F is malnormal in G and F is quasiconvex in G.*

Proof.

All nontrivial abelian subgroups of torsion-free hyperbolic groups are infinite cyclic [18] and therefore elementary. Thus Γ is non-abelian. Therefore Γ contains two non-commuting elements a_0, b_0. Notice that no nontrivial power of a_0 is equal to a power of b_0. Indeed, if $a_0^n = b_0^m$ for some $n > 0$ then the centralizer $C_G(a_0^n)$ of a_0^n contains both a_0 and b_0. However, centralizers of nontrivial elements in torsion-free hyperbolic groups are infinite cyclic [18] and therefore $C_G(a_0^n)$ is cyclic and abelian. This implies that a_0 and b_0 commute which contradicts our assumptions.

Therefore by Lemma 6.4 there exists $n > 0$ such that the subgroup $K = gp(a_0^n, b_0^n)$ is free of rank two and quasiconvex in G. Denote $a = a_0^n$, $b = b_0^n$ so that $K = gp(a,b) = F(a,b)$ is a free group on a,b.

By Lemma 6.5 the subgroup K coincides with its virtual normalizer. If K is malnormal in G then K satisfies all the requirements of Theorem 6.7 and there is nothing to prove. Suppose now that K is not malnormal in G. By Lemma 6.2 there exist only finitely many elements double cosets KzK where $z \in G, z \notin K$ is such that $zKz^{-1} \cap K \neq 1$. Let us denote these cosets $Kz_1K, \dots Kz_sK$.

Observe that if $z \notin K$ and $K \cap zKz^{-1} \neq 1$ then the subgroup $K_1 = K \cap zKz^{-1}$ has infinite index in both K and zKz^{-1}. Indeed, if both these indices are finite then $z \in V_G(K) = K$. Suppose that one of these indices is finite and the other is infinite. Without loss of generality (replace z by z^{-1} if necessary) we may assume that K_1 has finite index in K and infinite index in zKz^{-1}. Then by Lemma 6.6 for every $i > 0$ the subgroup $K_i = K \cap z^iKz^{-i}$ has finite index in K and infinite index in z^iKz^{-i}. In particular $K_i \neq 1$. By Lemma 6.3

this implies that $z^p \in K$ for some $p > 0$. However this means that $K_p = K = z^pKz^{-p} = K \cap z^pKz^{-p}$ has finite index (in fact index one) in z^pKz^{-p}. This gives us a contradiction. Thus we have established that for every $z \in G, z \notin K$ such that $K \cap zKz^{-1} \neq 1$, the subgroup $K \cap zKz^{-1}$ has infinite index in both K and zKz^{-1}. Notice also that since K and zKz^{-1} are quasiconvex in G, their intersection $K \cap zKz^{-1}$ is also quasiconvex and so finitely generated.

Thus all the subgroups $H_i = z_iKz_i^{-1} \cap K$ are finitely generated and of infinite index in K. By Theorem 5.16 there exists a subgroup F of K such that F is free of rank two, F is malnormal in K and no nontrivial element of F is conjugate in K to an element of $H_1 \cup \cdots \cup H_s$.

We claim that F is malnormal in G. Indeed, suppose not. Then there exist $f_1, f_2 \in F$, $f_1 \neq 1$ and $g \in G$, $g \notin F$ such that $gf_1g^{-1} = f_2$. Since F is malnormal in K, this implies $g \notin K$. Therefore g has the form $g = k_1z_ik_2$ for some $k_1, k_2 \in K$, $1 \leq i \leq s$. Hence

$$k_1z_ik_2f_1k_2^{-1}z_i^{-1}k_1^{-1} = f_2$$

and

$$z_i(k_2f_1k_2^{-1})z_i^{-1} = k_1^{-1}f_2k_1.$$

Therefore $k_1^{-1}f_2k_1 \in z_iKz_i^{-1} \cap K = H_i$. However $f_2 \neq 1, f_2 \in F$ and the subgroup F of K was chosen so that no nontrivial element of F is conjugate in K to an element of H_i. This gives us a contradiction. So F is indeed malnormal in G.

Any finitely generated subgroup in a finite rank free group is quasiconvex [30] and therefore F is quasiconvex in K. Since K is quasiconvex in G, this implies that F is quasiconvex in G. Thus $F \leq K \leq \Gamma$ is a free group of rank two that is malnormal and quasiconvex in G. Theorem 6.7 is proved.

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