TWO-GENERATED GROUPS ACTING ON TREES

ILYA KAPOVICH[1] AND RICHARD WEIDMANN

Abstract. We study 2-generated subgroups of groups that act on simplicial trees. We show that any generating pair $\{g, h\}$ of such a subgroup is Nielsen-equivalent to a pair $\{f, s\}$ where either powers of f and s or powers of f and sfs^{-1} have a common fixed point if the subgroup $\langle g, h\rangle$ is freely indecomposable. Analogous results are obtained for generating pairs of fundamental groups of graphs of groups. Some simple applications are given.

1. Introduction

A. Karrass and D. Solitar [8] studied subgroups of amalgamated products of type $G = A *_C B$ where $C \neq 1$ is proper and malnormal in A and B. They showed in particular that a 2-generated subgroup of G is either a free product of two cyclic groups or is conjugate to a subgroup of either A or B and that G cannot be generated by 2 elements. More recently S. Bleiler and A. Jones [2] obtained similar results for the case that the amalgamated subgroup C is malnormal in one of the factors. For HNN-extensions with malnormal associated subgroups S. Pride [10] studied two element generating systems (see also [12] and [9] in this context). This allowed S. Pride [11] to solve the isomorphism problem for two-generated one-relator groups with torsion.

All of these results can be proved using the Nielsen method for amalgamated products as developed by H. Zieschang [15] and refined in [5] and [14] and the Nielsen method for HNN-extensions due to N. Peczynski and W. Reiwer [12]. T. Delzant [6] also uses a geometric variation of Nielsen's method to prove that a torsion-free word-hyperbolic group contains only finitely many conjugacy classes of two-generated one-ended subgroups.

In this note we use related methods to study two-generator subgroups of fundamental groups of graphs of groups. Our approach is purely geometric, meaning that we study group actions on simplicial trees which is an equivalent point of view by the Bass-Serre theory (see [13] and [1] for details). We do not make any malnormality assumptions nor do we impose any other restrictions. The proof of the main result could also have been achieved using induction and extensions of the techniques mentioned before, but we believe that the present proof is more elegant and also more transparent.

The proof is very much in the spirit of the first chapters of [4] to which we refer the reader for some of the vocabulary. We state the main result of this note:

Theorem 1. *Let G be a group acting on a simplicial tree T without inversions such that the stabilizer of each edge is non-trivial. Let $g, h \in G$ such that either*

(i) $\langle g, h\rangle$ is not cyclic and not the free product of two cyclic groups or

(ii) $\langle g, h\rangle = G$, i.e. $\{g, h\}$ is a generating set of G.

Then $\{g, h\}$ is Nielsen equivalent to $\{f, s\}$ such that either

(1) Some non-trivial powers of f and s have a common fixed point or

(2) Some non-trivial powers of f and sfs^{-1} have a common fixed point.

1991 *Mathematics Subject Classification.* 20F32.

[1] The first author acknowledges the support of the Golda Meir post-doctoral fellowship at the Hebrew University of Jerusalem for the 1997/98 academic year

The assumption that the edge stabilizers are non-trivial is only needed to exclude the case that G itself is the free product of two cyclic groups. An immediate corollary of Theorem 1 is:

Corollary 1.1. *Let T be the Bass-Serre tree associated to the presentation of the two-generated group G as the fundamental group of a graph of groups with non-trivial edge groups. We consider the natural action of G on T. Then any generating pair $\{g, h\}$ is Nielsen equivalent to a pair $\{f, s\}$ (also generating G) such that one of the following holds:*
(1) Some nontrivial powers of f and s fix a common vertex.
(2) Some nontrivial powers of f and sfs^{-1} fix a common vertex.

In Section 2 we will prove Theorem 1; in section 3 we will conclude by showing how to use Corollary 1.1 to deduce (stronger versions of) the results mentioned in the beginning.

2. The proof of Theorem 1

A *simplicial tree* T is a connected and simply connected one-dimensional simplicial complex. We refer to the 0-simplices of T as *vertices* of T and to the 1-simplices of T as *edges* of T. (Note that T is not required to be a locally finite simplicial complex, that is a 0-simplex is allowed to be the face of infinitely many 1-simplices.)

Every simplicial tree T comes equipped with a natural metric d. For any two vertices v_1, v_2 of T we define $d(v_1, v_2)$ to be the minimal number of edges in an edge-path from v_1 to v_2. If we endow each edge with the metric of the unit interval $[0, 1] \subset \mathbb{R}$, then d naturally extends to a metric on T.

By an *automorphism* of T we mean an isometry of (T, d) which takes vertices to vertices (and therefore edges to edges). An automorphism ϕ of T is called an *inversion* if ϕ interchanges the endpoints of some edge e.

We will say that G *acts* on a simplicial tree T if G acts by automorphisms of T. The reader is referred to [13] and [1] for a more detailed and rigorous discussion of group actions on simplicial trees.

If G is a group and X is a subset of G, we denote by $\langle X\rangle$ the subgroup of G, generated by X.

Let G be a group acting on a simplicial tree T without inversions and let $g \in G$. By T_g we denote the set that consists of all points of T that are fixed under the action of a non-trivial power of g, i.e.

$$T_g = \{x \in T \mid g^z x = x \text{ for some } z \in \mathbb{Z} \text{ such that } g^z \neq 1 \}.$$

It is easy to see that T_g is a subtree of T for all $g \in G$: We clearly only need to verify that T_g is connected. If g is a hyperbolic element this is trivial since T_g is the empty set. If g is elliptic then there exists a $x \in T$ such that $gx = x$. This implies in particular that $x \in T_g$, it is further clear that $g^z x = x$ for all $z \in \mathbb{Z}$. Let now $y \in T_g$ and choose $z \in \mathbb{Z}$ such that $g^z y = y$. Now g^z fixes x and y and therefore the segment $[x, y]$. It follows that $[x, y] \subset T_g$. It follows that T_g is connected. It is further easy to see that $gT_g = T_g$ for all $g \in G$.

We can now reformulate Theorem 1 as:

Theorem 1A *Let G be a group acting on a simplicial tree T without inversions such that the stabilizer of each edge is non-trivial. Let $g, h \in G$ such that either*
(i) $\langle g, h\rangle$ is not cyclic and not the free product of two cyclic groups or
(ii) $\langle g, h\rangle = G$, i.e. $\{g, h\}$ is a generating set of G.
Then $\{g, h\}$ is Nielsen equivalent to $\{f, s\}$ such that either
(a) $T_f \cap T_s \neq \emptyset$ or
(b) $T_f \cap sT_f \neq \emptyset$.

We will first look at two special cases of Theorem 1A.

Lemma 2.1. *Let $f, s \in G$ be two elliptic elements such that $T_f \cap T_s = \emptyset$. Then*
(i) $\langle f, s\rangle = \langle f\rangle * \langle s\rangle$.
(ii) There exists an edge e of T such that $ge \neq e$ for all $g \in \langle f, s\rangle - 1$.

Proof. Suppose that $T_f \cap T_s = \emptyset$. Let $a \in T_f, b \in T_s$ be vertices of T such that $[a, b]$ is the bridge between T_f and T_s, that is $[a, b] \cap T_f = a$ and $[a, b] \cap T_s = b$. Then $d(a, b) = d(T_f, T_s) \geq 1$. Choose further c such that $[a, c]$ is an edge of T and that $[a, c] \subset [a, b]$.

To prove the lemma it is enough to show that any strictly alternating product of non-trivial powers of f and s does not fix $[a, c]$. For a non-trivial power of f this is immediate since $[a, c] \notin T_f$. We therefore only need to check elements of type

$$g = f^{n_1} s^{m_1} \ldots f^{n_k} s^{m_k}$$

where $k \geq 1$, $f^{n_i} \neq 1$ for $2 \leq i \leq k$ and $s^{m_i} \neq 1$ for $1 \leq i \leq k-1$ and $s^{m_1} \neq 1$ if $k = 1$. Put $p_0 = 1$ and for each $i = 1, \ldots, k$ put $p_i = f^{n_1} s^{m_1} \ldots f^{n_i} s^{m_i}$ and $q_i = p_{i-1} f^{n_i}$. We will show that $ga \neq a$ which implies that $g[a, c] = [ga, gc] \neq [a, c]$ since G acts without inversions.

Put $\gamma_i = [p_{i-1}a, q_i a] \cup [q_i a, q_i b] \cup [q_i b, p_i b] \cup [p_i b, p_i a]$ for $i = 1, \ldots, k-1$ and $i = k$ if $s^{m_k} \neq 1$ and $\gamma_i = [p_{i-1}a, q_i a] = [p_{i-1}a, p_i a]$ if $i = k$ and $s^{m_k} = 1$. Thus γ_i is a path from $p_{i-1}a$ to $p_i a$. Note, that γ_1 is a path from a to $p_1 a$.

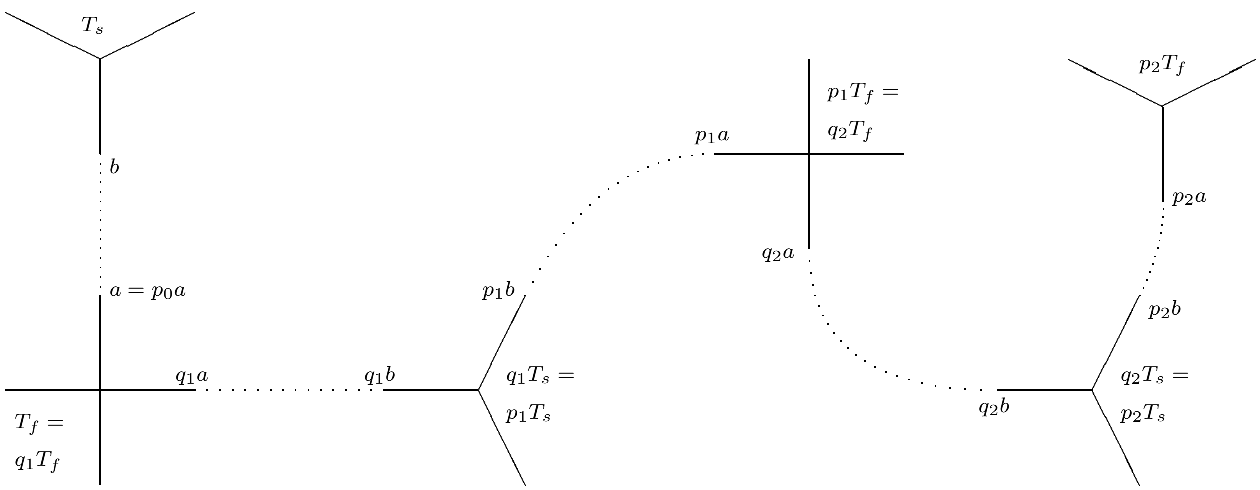

Figure 1

Claim.

(1) Each γ_i is a non-degenerate geodesic from $p_{i-1}a$ to $p_i a$, that is $d(p_{i-1}a, p_i a) = d(p_{i-1}a, q_i a) + d(q_i a, q_i b) + d(q_i b, p_i b) + d(p_i b, p_i a) > 0$ unless $i = k$ and $s^{m_k} = 1$.

(2) For every $i = 1, \ldots, k-1$ the path $\gamma_i \cup \gamma_{i+1}$ is a geodesic from $p_{i-1}a$ to $p_{i+1}a$ that is $\gamma_i \cap \gamma_{i+1} = p_i a$ and $d(p_{i-1}a, p_{i+1}a) = l(\gamma_i \cup \gamma_{i+1}) = l(\gamma_i) + l(\gamma_{i+1})$.

To prove part (1) of the claim, note first that $[p_{i-1}a, q_i a] \subseteq q_i T_f$. We know that $T_f \cap [a, b] = a$ and so $q_i T_f \cap [q_i a, q_i b] = q_i a$. Hence $[p_{i-1}a, q_i a] \cap [q_i a, q_i b] = q_i a$ and therefore $[p_{i-1}a, q_i a] \cup [q_i a, q_i b]$ is a geodesic in T ending in $[q_i a, q_i b]$. Thus, by pasting geodesics, to see that part (1) of the claim holds, it is enough to establish that $[q_i a, q_i b] \cup [q_i b, p_i b] \cup [p_i b, p_i a]$ is a geodesic with a nontrivial initial segment $[q_i a, q_i b]$ (see Figure 1).

To see this, suppose first that $q_i b \neq p_i b$. Note that $[q_i b, p_i b] \subseteq q_i T_s = p_i T_s$. Hence the definition of $[a, b]$ implies that $[q_i a, q_i b] \cap q_i T_s = q_i b$ and $p_i T_s \cap [p_i b, p_i a] = p_i b$. Thus all three segments $[q_i a, q_i b]$, $[q_i b, p_i b]$ and $[p_i b, p_i a]$ are non-degenerate and $[q_i a, q_i b] \cap [q_i b, p_i b] = q_i b$ and $[q_i b, p_i b] \cap [p_i b, p_i a] = p_i b$. Therefore $[q_i a, q_i b] \cup [q_i b, p_i b] \cup [p_i b, p_i a]$ is a geodesic with a non-trivial initial segment $[q_i a, q_i b]$ as required. (In Figure 1 only this case occurs.) Suppose now that $q_i b = p_i b = z$. Hence $b = q_i^{-1} q_i b = q_i^{-1} p_i b = s^{m_i} b$. If

$[z, q_i a] \cap [z, p_i a] = z$, the statement follows. If $[z, q_i a] \cap [z, p_i a] \neq z$ then choose (a vertex) $y \neq z$ of T such that $[z, q_i a] \cap [z, p_i a] = [z, y]$ (see Figure 2).

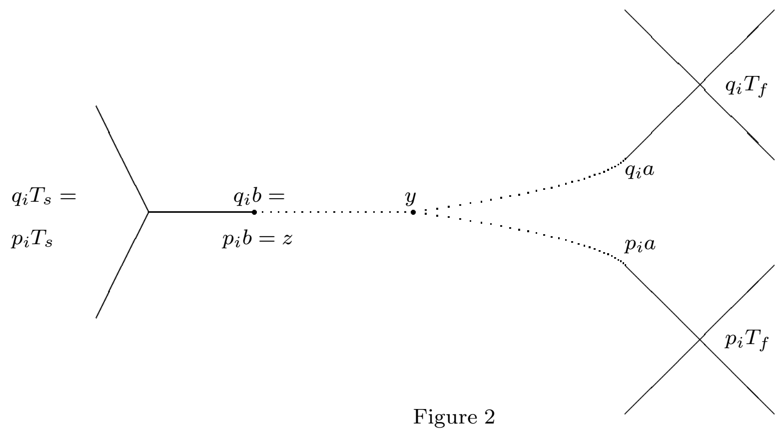

Figure 2

Since $p_i q_i^{-1} = q_i s^{m_i} q_i^{-1}$ maps $[z, q_i a]$ isometrically onto $[z, p_i a]$ we get that $q_i s^{m_i} q_i^{-1}$ fixes y. It follows that s^{m_i} fixes $q_i^{-1} y$. Note that $y \notin q_i T_s$ and therefore $q_i^{-1} y \notin T_s$, a contradiction to the definition of T_s. Thus $[q_i a, q_i b] \cup [q_i b, p_i b] \cup [p_i b, p_i a]$ is a geodesic with a nontrivial initial segment $[q_i a, q_i b]$ as required and part (1) of the claim is proved.

To see that part (2) of the claim holds, we need to show that

$$[p_{i-1} b, p_{i-1} a] \cap \big([p_{i-1} a, q_i a] \cup [q_i a, q_i b]\big) = p_{i-1} a$$

if not $i = k$ and $s^{m_k} = 1$ and $p_{i-1} b, p_{i-1} a] \cap [p_{i-1} a, q_i a] = p_{i-1} a$ otherwise. Note that $[p_{i-1} a, q_i a] \subseteq p_{i-1} T_f = q_i T_f$. If $p_{i-1} a \neq q_i a$ then the statement above follows from the fact that $[p_{i-1} b, p_{i-1} a] \cap p_{i-1} T_f = p_{i-1} a$. If $p_{i-1} a = q_i a = z$ then, similarly to part (1) we see that $[z, p_{i-1} b] \cap [z, q_i b] = z$. This implies the desired statement and completes the proof of the claim.

Note that for each $i = 1, \ldots, k$ we have $0 \neq l(\gamma_i) \geq 1$. Therefore the Claim implies that $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ is a geodesic between $p_0 a = a$ and $p_k a = ga$ and $d(a, ga) = d(p_0 a, p_k a) = l(\gamma_1) + \cdots + l(\gamma_k) \geq k > 0$. It follows that $ga \neq a$. This completes the proof of the lemma. $\square$

Lemma 2.2. *Let $f, s \in G$ where f is elliptic and s is hyperbolic. Assume that the translation length of s is minimal among the translation lengths of all elements of type $f^{z_1} s f^{z_2}$ with $z_1, z_2 \in \mathbb{Z}$ and that $T_f \cap s T_f = \emptyset$. Then*

(i) $\langle f, s \rangle = \langle f \rangle * \langle s \rangle \cong \langle f \rangle * \mathbb{Z}$.

(ii) There exists an edge e of T such that $ge \neq e$ for all $g \in \langle f, s \rangle - 1$.

Proof. Let C be the axis of s, l the translation length of s and a be a fixed point of f. We first show that the minimality of the translation length of s guarantees that any $x \in C$ for which $f^z x \in C$ holds is fixed under the action of f^z, i.e. that $f^z x = x$. This implies in particular that $C \cap f^z C \subset T_f$ if $f^z \neq 1$. Suppose there is a $x \in C$ such that $f^z x = y \in C$ for some $z \in \mathbb{Z}$ where $x \neq y$. It follows that f^z maps $[a, x]$ isometrically onto $[a, y]$. Choose z such that $[a, x] \cap [a, y] = [a, z]$. It is clear that f^z maps $[z, x]$ isometrically onto $[z, y]$ and that f^z fixes $[a, z]$ pointwise and therefore fixes z. Now $[x, z] \cup [z, y]$ is clearly the geodesic segment $[x, y]$ which implies that $[x, z] \cup [z, y] \subset C$ (See Figure 3). Continuity guarantees that for any $\lambda \in [0, d(x, y)]$ there is a point $r \in [x, z] \subset C$ such that $f^z r \in [z, y] \subset C$ and $d(r, f^z r) = \lambda$. Choose λ such that $0 < \lambda \leq l$ and r such that $d(r, f^z r) = \lambda$. It follows that either $d(r, s f^z r) = l - \lambda$ or

$d(f^z r, sf^{-z} f^z r) = d(f^z r, sr) = d(r, f^{-z} sr) = l - \lambda$, in both cases a contradiction to the minimality of the translation length of s.

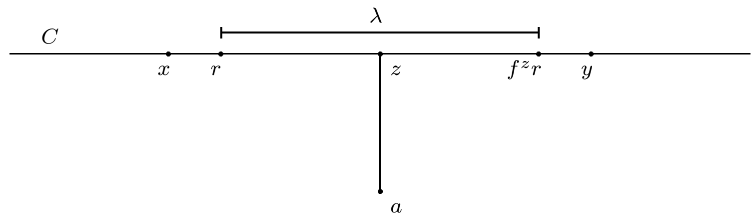

Figure 3

Suppose now that $T_f \cap sT_f = \emptyset$. It is easy to see that $T_f \cap s^z T_f = \emptyset$ for all $z \in \mathbb{Z} - 0$. Let a be a fixed point of f and $e = [b, c]$ be an edge of T such that $T_f \cap [b, c] = b$. We will show that no strictly alternating product of non-trivial powers of f and s fixes e. This clearly implies the assertion of the lemma. If g is a non-trivial power of f then g does not fix e since $e \notin T_f$. Let now g be a strictly alternating product of non-trivial powers of f and s which is not merely a power of f. We will show that $gT_f \cap T_f = \emptyset$. This implies that g does not fix b, it follows that g does not fix e since G acts without inversions. Note that $gT_f = gf^z T_f$ for all $z \in \mathbb{Z}$ and we can therefore restrict ourselves to alternating products that ends with a power of s, i.e. g that can be written as

$$g = f^{n_1} s^{m_1} \dots f^{n_k} s^{m_k}$$

where $m_i \neq 0$ for $1 \leq i \leq k$ and $f^{n_i} \neq 1$ for $2 \leq i \leq k$. We define $p_0 := 1$ and for $1 \leq i \leq k$ $p_i := f^{n_1} s^{m_1} \dots f^{n_i} s^{m_i}$ and $q_i := p_{i-1} f^{n_i}$. We will show that any segment $[p_{i-1}a, p_i a]$ has a non-trivial subsegment σ_i such that $\sigma_i \cap p_{i-1} T_f = \emptyset$, $\sigma_i \cap p_i T_f = \emptyset$, $\sigma_i \cap [p_{i-2}a, p_{i-1}a] = \emptyset$ (if $i \geq 2$) and $\sigma_i \cap [p_i a, p_{i+1} a] = \emptyset$ (if $i \leq k-1$). Now $\sigma_{k-1} \cap p_k T_f = \emptyset$ clearly implies that $\sigma_1 \cap p_k T_f = \emptyset$ (see Figure 4). Since we also have that $\sigma_1 \cap T_f = \emptyset$ and since σ_1 is part of the bridge between T_f and $p_k T_f$ this implies that $T_f \cap p_k T_f = \emptyset$.

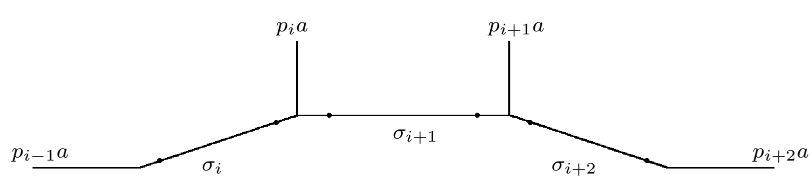

Figure 4

Note, that $p_{i-1}a$ lies in $p_{i-1}T_f = q_i T_f$ and is therefore fixed under the action of $p_{i-1} f^{n_i} p_{i-1}^{-1}$, which implies that $p_{i-1}a = p_{i-1} f^{n_{i+1}} p_{i-1}^{-1} p_{i-1} a = q_i a$. This means that $p_{i-1}a = q_i a$ gets mapped onto $p_i a$ under the action of $q_i s^{m_i} q_i^{-1} = p_i q_i^{-1}$, a hyperbolic element with axis $q_i C$. Now the geodesic segment $[p_{i-1}a, p_i a] = [q_i a, p_i a]$ clearly can be written as $b_i \cup [s_i, t_i] \cup c_i$ where $[s_i, t_i] := [p_{i-1}a, p_i a] \cap q_i C$ is a segment of length lm_i and where $b_i = [p_{i-1}a, s_i]$ and $c_i = [t_i, p_i a]$ are segments of length $d(p_{i-1}a, q_i C) = d(p_i a, q_i C)$. We define $\sigma_i := int([s_i, t_i] - (p_{i-1} T_f \cup p_i T_f))$. Note, that $\sigma_i \neq \emptyset$ since otherwise $p_{i-1} T_f \cap p_i T_f \neq \emptyset$ which implies that $q_i^{-1} p_{i-1} T_f \cap q_i^{-1} p_i T_f = f^{-n_i} T_f \cap s^{m_i} T_f = T_f \cap s^{m_i} T_f \neq \emptyset$ which contradicts our assumption. (See Figure 5)

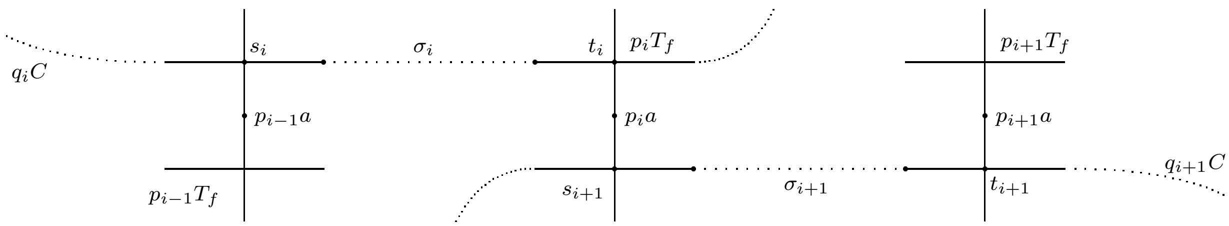

Figure 5

Suppose that $\sigma_i \cap [p_ia, p_{i+1}a] \neq \emptyset$ (the case $\sigma_i \cap [p_{i-2}a, p_{i-1}a] \neq \emptyset$ is analogous) for some $i \in \{1, \dots, k\}$. Choose $x \in \sigma_i$ such that $x \in [p_ia, t_{i+1}]$ or $x \in \sigma_{i+1}$ such that $x \in [s_i, p_ia]$. To see that such a x exists, pick an arbitrary $y \in \sigma_i \cap [p_ia, p_{i+1}a]$. If $y \in [p_ia, t_{i+1}]$ we can choose $x := y$ and are in the second case, otherwise $y \in [t_{i+1}, p_{i+1}a]$ which implies that $\sigma_{i+1} \subset [y, p_ia] \subset [s_i, p_ia]$ which guarantees that the first case occurs.

W.l.o.g. we can assume that the first case occurs, i.e. that $x \in \sigma_i$ such that $x \in [p_ia, t_{i+1}]$. Note, that $x \notin [p_ia, s_{i+1}] = b_{i+1}$ since $d(p_ia, x) > d(p_ia, q_iC) = d(p_ia, p_iC) = d(p_if^{n_{i+1}}p_i^{-1}p_ia, p_if^{n_{i+1}}p_i^{-1}p_iC) = d(p_ia, q_{i+1}C) = l(b_{i+1})$. It follows that $x \in [s_{i+1}, t_{i+1}]$ and therefore $x \in p_iC \cap q_{i+1}C$. This implies by the observation made in the beginning that $x \in p_iT_f$ which contradicts the definition of σ_i. The lemma is proved. □

Proof of Theorem 1A. We first verify that $\{g, h\}$ is Nielsen equivalent to a pair $\{f, s\}$ where f acts with a fixed point. We assume that there is no pair $\{f, s\}$ that is Nielsen equivalent to $\{g, h\}$ such that f (or s) acts with a fixed point, i.e. that for any such pair $\{f, s\}$ the translation lengths $l(f)$ and $l(s)$ are non-zero. Let now $\{f, s\}$ be a pair such that the translation lengths are minimal in the sense that $l(f^{z_1}sf^{z_2}) \geq l(s)$ and $l(s^{z_1}fs^{z_2}) \geq l(f)$ for $z_1, z_2 \in \mathbb{Z}$. Let C_1 and C_2 be the axes of f and s, respectively, and L the length of the arc $C_1 \cap C_2$ ($L = 0$ if C_1 and C_2 are disjoint). The minimality guarantees that $L \leq l(f)/2$ and $L \leq l(s)/2$. It follows that $L < min(l(f), l(s))$ which implies that $\langle f, s\rangle = \langle h_1, h_2\rangle$ is free of rank two and acts freely on T by Lemma 2.6 of [4]. This implies that neither (i) nor (ii) of the Theorem 1 can hold: Indeed, (i) obviously does not hold since the free group of rank two is the free product of two infinite cyclic groups. Also, (ii) does not hold since in Theorem 1 we assume that G-stabilizers of edges of T are nontrivial and so the action of G on T is not free.

Thus we only need to look at the case when f acts with a fixed point. We now replace s appropriately so that the translation length of s is minimal among the translation length of all elements of type $f^{z_1}sf^{z_2}$ with $z_1, z_2 \in \mathbb{Z}$. Note that if (ii) is fulfilled then every edge of T is fixed by a non-trivial element of $\langle f, s\rangle$. The statement of Theorem 1 now follows from Lemma 2.1 if s acts with a fixed point, i.e. if $T_s \neq \emptyset$, and from Lemma 2.2 otherwise. □

3. Some applications

In this section we provide some applications of Theorem 1 and of Corollary 1.1. We will say that subsets $M \subset G$ and $K \subset G$ are *equivalent* if there exists a set L such that M and L are Nielsen equivalent and that L and K are conjugate. Note that if M is a generating set of G that is equivalent to K then K also generates G. Following [7], we will call subgroups E and F of a group G *conjugacy separated* in G if $g^{-1}Eg \cap F = 1$ for all $g \in G$.

The following statement has been proven by S.Pride [10] for the case $A = A'$ (see [12] for an alternative proof). It should be noted that Pride's result does not make the assumption that A' and B are conjugacy separated, his result could however also easily be deduced from Corollary 1.1.

Corollary 3.1. *Let $G =< H, t, \mid t^{-1}A't = B >$ be a two-generated HNN-extension where A and B are non-trivial malnormal conjugacy separated subgroups of H and where A' is a normal subgroup of A. Then any generating pair is equivalent to a pair $\{a, th\}$ for some $a \in A$ and $h \in H$.*

Proof. Since A' and B are non-trivial, the given HNN-extension has non-trivial edge stabilizers and we can apply Corollary 1.1. If case (1) of Corollary 1.1 occurs, then G possesses a pair of generators which are both conjugate to elements of H. This implies that they are factored out by the epimorphism

$$\phi : G \to G/\langle\langle H\rangle\rangle \cong \mathbb{Z}, \text{ (where } \langle\langle H\rangle\rangle \text{ is the normal closure of } H \text{ in } G)$$

which is impossible since they were assumed to be a generating system for G.
Thus case (2) of Corollary 1.1 occurs. This means that there is a generating pair $\{f, s\}$ of G such that some nontrivial powers of f and sfs^{-1} fix a common vertex of T, that is $T_f \cap sT_f \neq \emptyset$. Let p be the vertex of T that is fixed under the action of H and let $q = tp$ be the point that is fixed under the action of tHt^{-1}. We will first show that, after conjugating the pair $\{f, s\}$, we can assume the following:
(i) $f \in A$ and $f^z \in A' - 1$ for some $z \in \mathbb{Z}$ and
(ii) $T_f = \bigcup_{a \in A} [p, aq]$.

Since f acts with a fixed point, we can conjugate the pair $\{f, s\}$ so that $f \in H$. Assume now that T_f is a single vertex, that is $T_f = p$. Since $T_f \cap sT_f \neq \emptyset$ we get $T_f \cap sT_f = T_f = p$ which implies that s fixes T_f and we are in case (1). Thus T_f contains an edge, i.e. there is a power $f^z \neq 1$ of f that fixes an edge emanating from p. So f^z lies in a conjugate of A' or B in H. If $f^z \in B$ then malnormality of B in H guarantees that $f \in B$. If $f^z \in A'$ we analogously conclude that $f \in A$. In both cases we can conjugate the pair $\{f, s\}$ so that $f \in A$ and $f^z \in A' - 1$ for some $z \in \mathbb{Z}$ which means that (i) holds.

To see that (ii) holds, it clearly suffices to show that every non-trivial element $a' \in A'$ fixes only the edges $[p, aq]$ where $a \in A$. (Indeed, it is obvious that a power of f, lying in $A - A'$, only fixes p since it is neither conjugate to an element of A' by an element of A since A' is normal in A, nor conjugate to an element of A' by an element of $H - A$ since A is malnormal in H, nor in H conjugate to an element of B since A and B are conjugacy separated.) It follows from the definition of the Bass-Serre tree T that a' fixes the edge $[p, q]$ whose stabilizer is A'. Therefore a' also fixes the edge $a[p, q] = [p, aq]$ since the stabilizer of this edge is $aA'a^{-1} = A'$ (recall that A' is normal in A). Now a' does not fix another edge emanating from aq since B has trivial intersection with all other conjugates of B in H (because B is malnormal in H) and since B has trivial intersection with all conjugates of A in H (because A and B are conjugacy separated in H). Similarly a' does not fix any edges emanating from p other then the edges $[p, aq]$. This proves (ii).

Now $T_f \cap sT_f \neq \emptyset$ and (ii) imply that either $s^\epsilon p = aq$ for some $\epsilon = \pm 1$ and $a \in A$ or that $s^\epsilon a_1 q = a_2 q$ for some $a_1, a_2 \in A$ and $\epsilon = \pm 1$. In the second case we get that $s^\epsilon = g a_2 a_1^{-1}$ where g fixes $a_2 q$. This implies that g is conjugate to an element of H and therefore $\phi(s^\epsilon) = \phi(g)\phi(a_2)\phi(a_1^{-1}) = 1$, i.e. that $\phi(s) = \phi(f) = 1$ which gives us a contradiction as in case (1). It follows that we can, possibly after replacing s by s^{-1}, assume that $sp = aq = atp$ for some $a \in A$. This implies that $s = atg$ where g fixes p and so belongs to H. Thus $s = ath$ with $h \in H$. We replace the generating pair $\{f, s\}$ by its conjugate $\{a^{-1}fa, a^{-1}sa = a^{-1}atha = t(ha) = th'\}$, where $a^{-1}fa \in A$ and $h' = ha \in H$. This completes the proof of Corollary 3.1. □

Note that the special case of Corollary 3.1, when A is infinite cyclic, was used in [9] to analyze the JSJ-decomposition of two-generated word-hyperbolic groups.

The following statement immediately implies the main result of S. Bleiler and A. Jones [2] mentioned before as well as the theorem of A. Karrass and D. Solitar [8] mentioned in the introduction.

Corollary 3.2. *Let $G = A *_C B$ be a two-generated group where $1 \neq C \subsetneqq A$, $1 \neq C \subsetneqq B$ and where C is malnormal in B. Then any generating pair of G is equivalent to a generating pair $\{f, s\}$ such that $f \in A$ and $f^z \in C - 1$ for some $z \in \mathbb{Z}$ and such that one of the following holds:*

(a) $s \in B - C$*;*
(b) $s = ab$ *with* $a \in A - C$ *and* $b \in B - C$ *and* $a^{-1}f^z a \in C - 1$ *for some* $z \in \mathbb{Z}$*;*
(c) $s = bab^{-1}$ *with* $a \in A - C$ *and* $b \in B - C$ *and* $a^z \in C - 1$ *for some* $z \in \mathbb{Z}$*.*

Proof. Let T be as in Corollary 1.1. Let p_A be the vertex of T that is fixed under the action of A and let p_B be the vertex that is fixed under the action of B.

Claim 1. We first prove that every elliptic element $g \in G - 1$ is conjugate to an element $h \in G$ such that

$$T_h \subset T_0 := \bigcup_{a \in A} [p_A, ap_B].$$

Note that T_0 is the union of all edges of T emanating from p_A. Since g is elliptic, it is conjugate to an element of A or B.

We first consider the case when g is conjugate to an element $h \in A$. Suppose that some nontrivial power $h^z \neq 1$ of h fixes a vertex $q \in T - T_0$. Then h^z also fixes the segment $[p_A, q]$, a segment of length at least two. Therefore h^z fixes two edges emanating from a vertex ap_B. This contradicts malnormality of C in B since the stabilizers of these edges are distinct conjugacy classes of aCa^{-1} in aBa^{-1} and thus have trivial intersection. Thus the statement of Claim 1 holds.

Suppose now that g is conjugate to an element h of B but not conjugate to an element of A (and therefore not conjugate to an element of C). Hence no nontrivial power of h is conjugate in B to an element of C since C is malnormal in B. Therefore no nontrivial power of h fixes an edge of T. This implies that $T_h = p_B \subset T_0$ and Claim 1 is proved.

The proof of Claim 1 shows in particular that if $g^n \neq 1$ fixes $[p_A, p_B]$ (that is $g^n \in C$) then $g \in A$. Besides, it shows that if $g^n \in A - 1$ then $g \in A$, i.e. that A is an isolated subgroup of G. It is further clear that a non-trivial power g^z of an element $g \in G$ can only fix a vertex $\bar{g}p_B$ with $\bar{g} \in G$ if g fixes either $\bar{g}p_B$ or a vertex joint to $\bar{g}p_B$ by a single edge.

Let $\{h_1, h_2\}$ be a generating pair for G. By Corollary 1.1 it is equivalent to a generating pair $\{f, s\}$ such that either (1) or (2) of Corollary 1.1 holds. We now show:

Claim 2. The pair $\{f, s\}$ can be conjugated (and if (1) occurs possibly interchanged) such that

$$(*) \qquad f \in A \text{ and } f^z \in C - 1 \text{ for some } z \in \mathbb{Z}.$$

Since $g^n \in C$ implies $g \in A$, to see this it suffices to show that a non-trivial power of f or s fixes an edge of T, i.e. that either T_f or T_s contains an edge. We assume that neither T_f nor T_s contains an edge. If case (1) of Corollary 1.1 occurs this implies that $T_f = T_s = T_f \cap T_s = \{v\}$ for some vertex $v \in T$. It follows that both f and s lie in a subgroup conjugate to either A or B, which contradicts the fact that $\{f, s\}$ is a generating set of G. If case (2) of Corollary 1.1 occurs, $sT_f \cap T_f \neq \emptyset$ implies that s fixes the single vertex of T_f. As before, this yields a contradiction. Thus Claim 2 is established. Note further that $(*)$ implies $T_f \subseteq T_0$.

It remains to show that s can be chosen to be of one of the types (a)-(c).
Assume first that case (1) of Corollary 1.1 occurs, that is $T_f \cap T_s \neq \emptyset$. If $p_A \in T_f \cap T_s$ then $f \in A$ and $s \in A$ since A is an isolated subgroup of G. This is impossible since f and s generate G.

Since $T_f \subseteq T_0$, it follows that $ap_B \in T_f \cap T_s$ for some $a \in A$. Thus $ap_B = f^n ap_B$ for some $f^n \neq 1$. Hence $p_B = a^{-1}f^n ap_B$ and $a^{-1}f^n a \in B$. Since $f \in A$, this means $a^{-1}f^n a \in C$. Thus we can conjugate the pair $\{f, s\}$ by a while preserving the condition of Claim 2. This allows us to assume that $a = 1$ and $p_B \in T_f \cap T_s$. If s fixes p_B then $s \in B$ and we have a generating pair $\{f, s\}$ which satisfies (a). If s does not fix p_B the we conclude by the observation made after the proof of Claim 1 that s fixes a vertex joined to p_B by an edge of T, that is a vertex of type bp_A for some $b \in B$. However, this implies that $s = bab^{-1}$ for some $a \in A$. It is further clear that $a^z \in C - 1$ for some $z \in \mathbb{Z}$ since otherwise p_B wouldn't lie in T_s. This implies that we are in situation (c).

Assume now that case (2) of Corollary 1.1 occurs, that is $sT_f \cap T_f \neq \emptyset$. Since the orbits of p_A and p_B under the action of G are disjoint and $T_f \subseteq T_0$, this implies that either $sp_A = p_A$ or that

$sap_B = a'p_B$ for two points ap_B and $a'p_B$ of T_f (with $a, a' \in A$). In the first case we conclude that s fixes p_A and therefore $s \in A$. As before this contradicts $\{f, s\}$ being a generating set. In the second case we have $sap_B = a'p_B$, $(a')^{-1}sap_B = p_B$ that is $(a')^{-1}sa = b \in B$ and $s = a'ba^{-1}$. Recall that $ap_B \in T_f$ and so $f^n ap_B = ap_B$ for some $f^n \neq 1$. Since $f \in A$, this means $a^{-1}f^n a \in B \cap A = C$. Hence we can conjugate the pair $\{f, s = a'ba^{-1}\}$ by a while preserving condition $(*)$ of Claim 2. We denote the resulting generating pair $\{a^{-1}fa, a^{-1}a'b\}$ by $\{f', s'\}$. To see that it falls into case (b) we recall that $f^z \neq 1$ fixes $a'p_B$ for some $z \in \mathbb{Z}$ which implies that f'^z fixes $a^{-1}a'p_B$. It follows that ${s'}^{-1}{f'}^z s' p_B = {s'}^{-1}{f'}^z a^{-1}a'p_B = {s'}^{-1}a^{-1}a'p_B = p_B$ which implies that ${s'}^{-1}{f'}^z s' \in B$. This can clearly only happen if in the product ${s'}^{-1}{f'}^z s' = b^{-1}{a'}^{-1}aa^{-1}f^z aa^{-1}a'b$ the subword ${a'}^{-1}aa^{-1}f^z aa^{-1}a'$ lies in C which is exactly the second statement of (b). This completes the proof of Corollary 3.2. ∎

References

[1] H. Bass, *Covering theory for graphs of groups,* J. Pure and Appl. Algebra, **89**, 1993, no. 1-2, 3–47

[2] S.Bleiler and A. Jones, *The free product of groups with amalgamated subgroup malnormal in a single factor*, J.Pure and Appl. Algebra **127**, 1998, 119-136.

[3] S.Bleiler and A. Jones, *On Two-Generator satellite knots*, preprint

[4] M. Culler and J. Morgan, *Group actions on* **R**-*trees*, Proc. London Math. Soc. (3), **55**, no. 3, 1987, 571–604

[5] D. Collins and H. Zieschang, *On the Nielsen method in free products with amalgamated subgroups*, Math. Z., **197**, 1987, 97–118

[6] T. Delzant, *Sous-groupes a deux generateurs des groups hyperboliques*, in " Group theory from a geometric viewpoint", Proc. ICTP. Trieste, World Scientific, Singapore, 1991, 177–192

[7] O. Kharlampovich and A. Myasnikov, *Hyperbolic groups and free constructions*, Transact. Amer. Math. Soc., **350**, 1998, no. 2, 571–613

[8] A. Karrass and D. Solitar, *The free product of two groups with a malnormal amalgamated subgroup*, Can. J. Math., **6**, 1971, 933-959

[9] I.Kapovich and R.Weidmann, *On the structure of two-generated hyperbolic groups*, to appear in Math. Z.

[10] S. Pride, *On the generation of one-relator groups*, Trans. Am. Math. Soc., **210**, 1975, 331-363

[11] S. Pride, *The isomorphism problem for two-generator one-relator groups with torsion is solvable,* Trans. Amer. Math. Soc. **227**, 1977, 109–139.

[12] N. Peczynski and W. Reiwer, *On cancellations in HNN-groups* Math. Z., **158**, 1978, 79–86

[13] J. P. Serre, *Trees*, Springer-Verlag, New York, 1980

[14] R. Weidmann, *Über den Rang von amalgamierten Produkten und NEC-Gruppen*, Dissertation, Ruhr-Universität Bochum, 1997

[15] H. Zieschang, *Über die Nielsensche Kürzungsmethode in freien Produkten mit Amalgam*, Invent. Math., **10**, 1970, 4–37

Dept. of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
E-mail address: `ilia@math.rutgers.edu`

Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
E-mail address: `Richard.Weidmann@ruhr-uni-bochum.de`