ON GROUP-THEORETIC MODELS OF RANDOMNESS AND GENERICITY

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Abstract. We compare the random group model of Gromov and the model of generic groups of Arzhantseva and Ol'shanskii.

1. Introduction

In recent years the study of randomness and genericity in group theory has been the subject of very active study. For example, Gromov [9] used probabilistic methods to prove the existence of a finitely generated group that does not admit a uniform embedding into a Hilbert space. Gromov's "density model" of random groups has been further explored by Ollivier [15, 16, 17, 18], Zuk [20], Ghys [7] and others. An alternative model of generic groups was defined by Arzhantseva and Ol'shanskii [5]. A number of results related to that model were obtained in [1, 2, 3, 4, 11, 12, 13]. For example, Kapovich, Schupp and Shpilrain [12] discovered a phenomenon of Mostow-type isomorphism rigidity for generic one-relator groups using the Arzhantseva-Ol'shanskii model.

Our goal in this note is to clarify the relationship between these two models. While there is no direct connection between them, it turns out that proofs using the Arzhantseva-Ol'shanskii model often imply that a certain variation of Gromov's density randomness condition holds.

Recall that in the random group model introduced by Gromov one first *fixes* a density parameter $0 < d < 1$. Then, given a number of generators $k \geq 2$ and an integer $n >> 1$, from the set of all cyclically reduced words of length n in $F(a_1, \dots, a_k)$ one chooses uniformly randomly and independently $(2k-1)^{dn}$ elements forming a set R. The group

$$G = \langle a_1, \dots, a_k | \ R \ \rangle$$

is termed a *random group with density parameter* d or d*-random group*. One then tries to understand the properties of G as $n \to \infty$. Note that the number of defining relators $(2k-1)^{dn}$ grows exponentially in the length n of the relators. Also, crucially, the density parameter d does not depend on the number of generators k of G.

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On the other hand, in the Arzhantseva-Ol'shanskii model, both the number of generators $k \geq 2$ and the number of defining relators $m \geq 1$ are *fixed*. For $n >> 1$ one chooses uniformly randomly and independently m cyclically reduced words $r_1, \dots, r_m$ of length n and forms the group

$$G = \langle a_1, \dots, a_k | r_1, \dots, r_m \rangle.$$

The group G is a *generic* k-generator m-relator group and one tries to study the properties of G as $n \to \infty$.

For the purposes of comparison we need to introduce a variant of Gromov's model of randomness where the density parameter $d = d(k)$ depends on the number of generators k and where it is possible that $d(k) \to 0$ as $k \to \infty$. We call this notion *low-density randomness* (see Section 2 for precise definitions). We show in Theorem 4.2 that many algebraic genericity results obtained in the Arzhantseva-Ol'shanskii model do yield low-density random properties:

Theorem 1.1. *The following properties are monotone low-density random (where k varies over $k = 1, 2, \dots$):*

(1) [9, 16] *the property that a finite group presentation defines a group G that is one-ended, torsion-free and word-hyperbolic (in fact, this property is monotone random in Gromov's original sense* [9, 16]*).*
(2) *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G such that all $(k-1)$-generated subgroups are free and quasiconvex in G;*
(3) *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G with $rk(G) = k$.*
(4) *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G such that all L_k-generated subgroups of infinite index in G are free and quasiconvex in G (here L_k is any sequence of positive integers).*
(5) *the property that for a k-generated finitely presented group G such that there is exactly one Nielsen-equivalence class of k-tuples of elements generating non-free subgroups.*

Recall that for a finitely generated group G the *rank* of G, denoted $rk(G)$, is the smallest cardinality of a generating set for G.

It turns out that in many cases various properties that are generic in the Arzhantseva-Ol'shanskii model are not d-random in the sense of Gromov with d independent of k. Some key information for estimating the density parameter d in Gromov's model is contained in the *genericity entropy* of exponentially generic sets of cyclically reduced words in $F(a_1, \dots, a_k)$. The definition of exponential genericity for subsets of $F(a_1, \dots, a_k)$ requires that certain fractions converge to 1 exponentially fast as $n \to \infty$. Genericity entropy quantifies this convergence rate.

We observe here that, unlike the standard small cancellation conditions, for the Arzhantseva-Ol'shanskii "non-readability condition" the genericity entropy depends on the number of generators k and in fact converges to 1 as $k \to \infty$. This implies that, when translated into the language of Gromov's density model, various results using the Arzhantseva-Ol'shanskii model yield properties that are low-density random but which are not d-random for any fixed $d > 0$ which is independent of k. We prove this fact in detail (see Corollary 5.3 below) for the Arzhantseva-Ol'shanskii μ-non-readability condition.

We also prove (see Proposition 5.5 below) that the property for a finite presentation on k generators to define a group G with $rk(G) = k$ is low-density random but not d-random for any $d > 0$ independent of k. The same is true (see Corollary 5.4 below) for the analog of Magnus' Freiheitssatz, that is, for the property that for a group G defined by a finite presentation on the generators $a_1, \dots, a_k$ any proper subset of $a_1, \dots, a_k$ freely generates a subgroup of G.

We show, however, that certain results obtained in the Arzhantseva-Ol'shanskii genericity model do yield d-random properties in Gromov's sense. Thus we prove (see Theorem 6.6 below):

Theorem 1.2. *For any* fixed *integer $L \geq 2$ there is some $d_L > 0$ such that the property that all L-generated subgroups of infinite index in finitely presented group G are free is monotone d_L-random.*

2. The density model and low-density random groups

In this section we want to give some precise definitions and notation related to Gromov's density model.

Notation 2.1. For $k \geq 2$ let $\mathcal{C}_k \subseteq F(a_1, \dots, a_k)$ be the set of all cyclically reduced words in $F(a_1, \dots, a_k)$. If $\mathcal{P}_k \subseteq \mathcal{C}_k$, we denote $\overline{\mathcal{P}_k} := \mathcal{C}_k - \mathcal{P}_k$. For a subset $\mathcal{Q}_k \subseteq F(a_1, \dots, a_k)$ denote by $\gamma(n, \mathcal{Q}_k)$ the number of elements of length n in $\mathcal{Q}_k$.

Definition 2.2 (Random groups in the density model)**.** Let $\mathcal{G}$ be a property of finite presentations of groups. Let $0 < d < 1$.

We say that the property $\mathcal{G}$ is *random with density parameter d* if for every $k \geq 2$

$$\lim_{n\to\infty} \frac{R_k(n, d, \mathcal{G})}{\gamma(n, \mathcal{C}_k)^{m_n}} = 1,$$

were $m_n = (2k-1)^{dn}$ and $R_k(n, d, \mathcal{G})$ is the number of all m_n-tuples $(r_1, \dots, r_{m_n})$ of cyclically reduced words of length n such that the group with presentation

$$\langle a_1, \dots, a_k | r_1, \dots, r_{m_n} \rangle$$

has property $\mathcal{G}$.

We say that $\mathcal{G}$ is *monotone d-random* if for every $0 < d' \leq d$ the property $\mathcal{G}$ is d'-random.

Note that $\gamma(n, \mathcal{C}_k)^{m_n}$ is exactly the number of all presentations

$$\langle a_1, \dots, a_k | r_1, \dots, r_{m_n} \rangle$$

where the r_i are cyclically reduced words of length n.

Definition 2.3 (Low-density random groups)**.** We can consider a property $\mathcal{G}$ of finite presentations as $\mathcal{G} = (\mathcal{G}_k)_{k\ge 2}$ where for every $k \ge 2$ $\mathcal{G}_k$ is a property of finite group presentations on k generators $a_1, \dots, a_k$.

For every integer $k \ge 2$ let $0 < d(k) < 1$. We say that $\mathcal{G}$ is *low-density random with density sequence* $(d(k))_{k\ge 2}$ if for every integer $k \ge 2$ we have

$$\lim_{n\to\infty} \frac{R_k(n, d(k), \mathcal{G}_k)}{\gamma(n, \mathcal{C}_k)^{m_n}} = 1,$$

were $m_n = (2k-1)^{nd(k)}$ and $R_k(n, d(k), \mathcal{G}_k)$ is the number of all m_n-tuples $(r_1, \dots, r_{m_n})$ of cyclically reduced words of length n such that the group

$$\langle a_1, \dots, a_k | r_1, \dots, r_{m_n} \rangle$$

has property $\mathcal{G}_k$.

We say that $\mathcal{G}$ is *monotone low-density random with density sequence* $(d(k))_{k\ge 2}$ if for any sequence $(d'(k))_{k\ge 2}$ satisfying $0 < d'(k) \le d(k)$ the property $\mathcal{G}$ is low-density random with density sequence $(d'(k))_{k\ge 2}$.

Remark 2.4. In the above definition let $d := \inf_k d(k)$ and let $\mathcal{G}$ be monotone low-density random with density sequence $(d(k))_{k\ge 2}$. If $d > 0$ then $\mathcal{G}$ is d-random in the sense of Definition 2.2.

The situation where $d = 0$ does not, however, correspond to a special case of Definition 2.2.

In this paper we concentrate on monotone random and monotone low-density random properties. There are, however, important examples of non-monotone random properties. Thus Zuk proved [20] that Kazhdan's Property (T) is d-random for every $1/3 < d < 1/2$ but it is not d-random for $0 < d < 1/3$.

Note that if $\mathcal{G}$ is a monotone low-density random property and $\mathcal{G}'$ is a monotone random property with a density parameter $d > 0$ independent of k then $\mathcal{G} \cap \mathcal{G}'$ is again monotone low-density random. Moreover, the intersection of two monotone low-density random properties is also monotone low-density random.

3. Genericity entropy and low-density randomness

We recall the basic notion of genericity in the Arzhantseva-Ol'shanskii approach.

Definition 3.1. A subset $\mathcal{P}_k \subseteq \mathcal{C}_k$ is *exponentially generic* if

$$\lim_{n\to\infty} \frac{\gamma(n, \mathcal{P}_k)}{\gamma(n, \mathcal{C}_k)} = 1$$

and the convergence is exponentially fast, that is, there exist $a > 0$ and $0 < \sigma < 1$ such that for all $n \geq 1$

$$\frac{\gamma(n, \overline{\mathcal{P}_k})}{\gamma(n, \mathcal{C}_k)} \leq a\sigma^n.$$

This condition is equivalent to the fact that for some $0 < t < 1$ and some $c > 0$ we have:

$$(\dagger) \qquad \gamma(n, \overline{\mathcal{P}_k}) \leq c(2k-1)^{tn}, \text{ for all } n \geq 1.$$

Definition 3.2. Let $\mathcal{P}_k \subseteq \mathcal{C}_k$ be a set of cyclically reduced words. We define the *genericity entropy* $t = t(\mathcal{P}_k)$ of $\mathcal{P}_k$ as:

$$t := \limsup_{n\to\infty} \frac{\log \gamma(n, \overline{\mathcal{P}_k})}{n \log(2k-1)}.$$

We also define the *lower genericity entropy* $t' = t'(\mathcal{P}_k)$ as

$$t' := \liminf_{n\to\infty} \frac{\log \gamma(n, \overline{\mathcal{P}_k})}{n \log(2k-1)}.$$

It is easy to see that we always have $0 \leq t'(\mathcal{P}_k) \leq t(\mathcal{P}_k) \leq 1$ and that $\mathcal{P}_k \subseteq \mathcal{C}_k$ is exponentially generic if and only if $t(\mathcal{P}_k) < 1$.

A simple but crucial computation shows that genericity entropy controls the density parameter in Gromov's model of random groups:

Proposition 3.3. *Let $k \geq 2$ and let $\mathcal{P}_k \subseteq \mathcal{C}_k$.*

(1) *Suppose that $t := t(\mathcal{P}_k) < 1$. Let $0 < d < 1$ be such that $d < 1 - t$. Then:*

$$\lim_{n\to\infty} \frac{\#\ (2k-1)^{dn}\text{-tuples of elements of } \mathcal{P}_k \text{ of length } n}{\#\ (2k-1)^{dn}\text{-tuples of elements of } \mathcal{C}_k \text{ of length } n} = 1.$$

(2) *Suppose that $d > 1 - t'$ where $t' = t'(\mathcal{P}_k)$. Then*

$$\lim_{n\to\infty} \frac{\#\ (2k-1)^{dn}\text{-tuples of elements of } \mathcal{P}_k \text{ of length } n}{\#\ (2k-1)^{dn}\text{-tuples of elements of } \mathcal{C}_k \text{ of length } n} = 0.$$

Proof. (1) Recall that there exist $0 < c_0 < c_1 < \infty$ such that for every $n \geq 1$ we have

$$c_0(2k-1)^n \leq \gamma(n, \mathcal{C}_k) \leq c_1(2k-1)^n.$$

Indeed, a result of Rivin [19] shows that

$$\gamma(n, \mathcal{C}_k) = (2k-1)^n + 1 + (k-1)[1 + (-1)^n]$$

Thus for a fixed $k \geq 2$ we have $\gamma(n, \mathcal{C}_k) \sim (2k-1)^n$ where $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

Let $m = (2k-1)^{dn}$. The number N of m-tuples of elements of $\mathcal{C}$ of length n where at least one element does not belong to $\mathcal{P}_k$ satisfies

$$\frac{N}{\gamma(n,\mathcal{C}_k)^m} \le \frac{m\gamma(n,\overline{\mathcal{P}_k})\gamma(n,\mathcal{C}_k)^{m-1}}{\gamma(n,\mathcal{C}_k)^m} = \frac{m\gamma(n,\overline{\mathcal{P}_k})}{\gamma(n,\mathcal{C}_k)} \le$$
$$\le \frac{(2k-1)^{dn}c(2k-1)^{tn}}{c_0(2k-1)^n} = \frac{c(2k-1)^{(t+d)n}}{c_0(2k-1)^n} \to_{n\to\infty} 0.$$

This implies part (1) of the proposition.

(2) Again let $m = (2k-1)^{nd}$. Recall that $d > 1 - t'$, so that $t' > 1 - d$. Let t'' be such that $t' > t'' > 1 - d$. Then for $n >> 1$ we have

$$\gamma(n,\overline{\mathcal{P}_k}) \ge (2k-1)^{nt''}$$

and hence

$$\gamma(n,\mathcal{P}_k) = \gamma(n,\mathcal{C}_k) - \gamma(n,\overline{\mathcal{P}_k}) \le \gamma(n,\mathcal{C}_k) - (2k-1)^{nt''}.$$

Thus $\gamma(n,\mathcal{P}_k)^m$ is the number of m-tuples of elements of $\mathcal{P}_k$ of length n and it satisfies:

$$\frac{\gamma(n,\mathcal{P}_k)^m}{\gamma(n,\mathcal{C}_k)^m} \le \frac{(\gamma(n,\mathcal{C}_k) - (2k-1)^{nt''})^m}{\gamma(n,\mathcal{C}_k)^m} =$$
$$\left(\frac{\gamma(n,\mathcal{C}_k) - (2k-1)^{nt''}}{\gamma(n,\mathcal{C}_k)}\right)^m = \left(1 - \frac{(2k-1)^{nt''}}{\gamma(n,\mathcal{C}_k)}\right)^m$$

Denote $Y_n = \log \frac{\gamma(n,\mathcal{P}_k)^m}{\gamma(n,\mathcal{C}_k)^m}$. Then

$$Y_n \le m \log\left(1 - \frac{(2k-1)^{nt''}}{\gamma(n,\mathcal{C}_k)}\right) = (2k-1)^{nd}\log\left(1 - \frac{(2k-1)^{nt''}}{\gamma(n,\mathcal{C}_k)}\right) \sim$$
$$(2k-1)^{nd}\left(-\frac{(2k-1)^{nt''}}{\gamma(n,\mathcal{C}_k)}\right) \sim -(2k-1)^{nd}\frac{(2k-1)^{nt''}}{(2k-1)^n} =$$
$$= -\frac{(2k-1)^{n(d+t'')}}{(2k-1)^n} = -\left(\frac{(2k-1)^{d+t''}}{2k-1}\right)^n \to_{n\to\infty} -\infty,$$

since $d + t'' > 1$. (Recall that $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.)

Hence $\lim_{n\to\infty} \log \frac{\gamma(n,\mathcal{P}_k)^m}{\gamma(n,\mathcal{C}_k)^m} = -\infty$ and therefore $\lim_{n\to\infty} \frac{\gamma(n,\mathcal{P}_k)^m}{\gamma(n,\mathcal{C}_k)^m} = 0$, as claimed.

□

Corollary 3.4. *For each $k \ge 2$ let $\mathcal{P}_k \subseteq \mathcal{C}_k$. Let $\mathcal{G} = (\mathcal{G}_k)_{k\ge 2}$ where $\mathcal{G}_k$ is the property that for a finite presentation on k generators all the defining relations belong to $\mathcal{P}_k$. Let $t_k = t(\mathcal{P}_k)$ and let $t'_k = t'(\mathcal{P}_k)$. Then the following hold:*

(1) *If* $0 \le t_k < 1$ *for every* $k \ge 2$ *then the property* $\mathcal{G}$ *is monotone low-density random.*
(2) *If* $\sup_k t'_k = 1$ *then there does not exist* $d > 0$ *such that* $\mathcal{G}$ *is* d*-random.*

4. Comparing the two models

The proofs of most existing results related to the Arzhantseva-Ol'shanskii genericity model rely on proving that certain subsets $\mathcal{P}_k \subseteq \mathcal{C}_k$ are exponentially generic:

Proposition 4.1. [5, 1]
Let $0 < \lambda < 1$. *Let* $\mathcal{P}_k$ *be the set of all* λ*-non-readable words in* $\mathcal{C}_k$ *and let* $\mathcal{Q}_k$ *be the set of* (λ, L)*-non-readable words in* $\mathcal{C}_k$*, where* $L = L_k \ge 2$*. Then* $\mathcal{P}_k$ *and* $\mathcal{Q}_k$ *are exponentially generic in* $\mathcal{C}_k$*.*

We give the definition of μ-readability in Section 5 and the definition of (μ, L)-readability in Section 6 below.

Unlike the case of the standard small cancellation condition, the genericity entropy t for exponentially generic sets from Proposition 4.1 depends on k.

A careful analysis of the proofs of exponential genericity of both $\mathcal{P}_k$ and $\mathcal{Q}_k$ shows that to prove this genericity one obtains upper bounds establishing that $t(\mathcal{P}_k) \le z(k) < 1$ with $\lim_{k\to\infty} z(k) = 1$. This situation is different from the standard small cancellation conditions where the genericity entropy is easily seen to have a positive upper bound which is separated from 1 and independent of k. Moreover, it is often the case that in these situations $\lim_{k\to\infty} t'(\mathcal{P}_k) = \lim_{k\to\infty} t(\mathcal{P}_k) = 1$.

We demonstrate this in Section 5 below for the Arzhantseva-Ol'shanskii μ-non-readability condition. Hence for $0 < d(k) < 1 - t'(\mathcal{P}_k)$ we have $\lim_{k\to\infty} d(k) = 0$ and, in view of Corollary 3.4, the notion of low-density randomness becomes necessary.

In most results related to the Arzhantseva-Ol'shanskii genericity one works with intersections of properties that either low-density random (such as conditions involving λ-non-readable words and (λ, L)-non-readable words) or random with some density parameter $d > 0$ independent of k (such as the small cancellation condition $C'(\lambda)$ for a fixed $\lambda < 1$). Therefore the resulting conditions are in fact low-density random. We give here a summary of some statements that follow from the proofs of various known results related to Arzhantseva-Ol'shanskii genericity using Corollary 3.4. Next to each item we give a reference to the source where the corresponding statement was established in the Arzhantseva-Ol'shanskii model of genericity.

Theorem 4.2. *The following properties are monotone low-density random (where* k *varies over* $k = 1, 2, \dots$*):*

(1) [5] *the property that a finite group presentation defines a group* G *that is one-ended, torsion-free and word-hyperbolic and satisfies the* $C'(\lambda)$*-small cancellation condition (where* $0 < \lambda \le 1/6$ *is any fixed number independent of* k*).*

(2) [5] *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G such that all $(k-1)$-generated subgroups are free and quasiconvex in G;*
(3) [5] *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G with $rk(G) = k$.*
(4) [1] *the property that a finite presentation on generators $a_1, \dots, a_k$ defines a group G such that all L_k-generated subgroups of infinite index in G are free and quasiconvex in G (here L_k is any sequence of positive integers).*
(5) [11] *the property that for a k-generated finitely presented group G such that there is exactly one Nielsen-equivalence class of k-tuples of elements generating non-free subgroups.*

Note that condition (1) in Theorem 4.2 is in fact monotone random and not just low-density random [9, 16].

Problem 4.3. As an explicit computation in Section 5 below shows, the actual combinatorial genericity conditions necessary to guarantee the algebraic conclusions of Theorem 4.2 have genericity entropy $t'(k) = t(k) \to 1$ as $k \to \infty$. Hence the corresponding combinatorial conditions in Gromov's approach are low-density random but are not d-random for any $d > 0$ independent of k. We also prove in Proposition 5.5 below that the property for a presentation on k generators to define a group of rank k is not d-random for any $d > 0$ independent of k.

It would be interesting to know if conditions (2), (4) and (5) from Theorem 4.2 are d-random for some $d > 0$ which is independent of k.

We suspect that the answer in most cases is negative, as indicated by Corollary 5.4 and Proposition 5.5.

5. Detailed examples

We recall the definition of one of the key genericity conditions for many results using the Arzhantseva-Ol'shanskii genericity.

Definition 5.1. [5] Let $0 < \mu < 1$ and let $k \geq 2$ be an integer. A freely reduced word w in $F(a_1, \dots, a_k)$ is *μ-readable* if there exists a finite connected graph Γ with the following properties:

(1) Every edge e of Γ is labelled by some element $s(e)$ of $\{a_1, \dots, a_k\}^{\pm 1}$ so that for every edge e we have $s(e^{-1}) = s(e)^{-1}$.
(2) The graph Γ is *folded* that is, there is no vertex with two distinct edges originating at that vertex and having the same label.
(3) The fundamental group of Γ is free of rank at most $k-1$.
(4) There exists an immersed path in Γ labelled w.
(5) The volume of Γ (that is, the number of non-oriented edges) is at most $\mu|w|$.

A key result of [5] is that for fixed k and a sufficiently small μ (namely, when $\mu < \log_{2k}\left(1+\frac{1}{4k-4}\right)$) the set of μ-readable elements of F_k is exponentially negligible in $\mathcal{C}_k$. However, the genericity entropy depends on k in this case and in fact converges to 1 as $k \to \infty$:

Proposition 5.2. *Let let $k \geq 2$ and let $0 < \mu_k < 1$. Let $\mathcal{P}_k \subseteq \mathcal{C}_k$ be the set of all cyclically reduced words in $F(a_1, \dots, a_k)$ that are not μ_k-readable. Then*

$$1 \geq t(\mathcal{P}_k) \geq t'(\mathcal{P}_k) \geq \frac{\log(2k-3)}{\log(2k-1)}$$

and hence

$$\lim_{k\to\infty} t(\mathcal{P}_k) = \lim_{k\to\infty} t'(\mathcal{P}_k) = 1.$$

Proof. Let Γ be the wedge of $k-1$ loop-edged labelled by $a_1, \dots, a_{k-1}$. Then any freely reduced word from $F(a_1, \dots, a_{k-1})$ can be read as the label of a path in Γ. Hence for any $w \in \mathcal{C}_{k-1}$ with $|w| > \mu_k(k-1)$ the word w is μ_k-readable, that is, $w \in \overline{\mathcal{P}_k}$. Thus for $n \geq 1 + \mu_k(k-1)$ we have

$$\gamma(n, \overline{\mathcal{P}_k}) \geq \gamma(n, \mathcal{C}_{k-1}) \geq c(2k-3)^n$$

for some constant $c > 0$ independent of n.

Therefore

$$t'(\mathcal{P}_k) \geq \liminf_{n\to\infty} \frac{\log c(2k-3)^n}{n\log(2k-1)} = \frac{\log(2k-3)}{\log(2k-1)},$$

as claimed. ☐

Part (2) of Corollary 3.4 immediately implies:

Corollary 5.3. *Let $0 < \mu_k < 1$ for $k \geq 2$. Let $\mathcal{G} = (\mathcal{G}_k)_{k\geq 2}$, where $\mathcal{G}_k$ is the property that for a finite presentation on k generators all the defining relations are non-μ_k-readable.*

Then $\mathcal{G}$ is not d-random for any $d > 0$.

Similarly, one obtains:

Corollary 5.4. *Let $\mathcal{G} = (\mathcal{G}_k)_{k\geq 2}$ where $\mathcal{G}_k$ is the property that a finite group presentation on the generators $a_1, \dots a_k$ defines a group G such that every proper subset of $a_1, \dots a_k$ freely generates a free subgroup of G.*

Then $\mathcal{G}$ is monotone low-density random but not d-random for any $d > 0$.

Proof. The fact that $\mathcal{G}$ is not d-random for any $d > 0$ follows from part (2) of Corollary 3.4 by the same argument as in the proof of Proposition 5.2.

It is well-known [9, 16] that the $C'(1/6)$ small cancellation condition is a monotone random property. It is also easy to see that the set of cyclically reduced words r in $\mathcal{C}_k$ such that every subword of r of length $|r|/6$ involves all the generators $a_1, \dots, a_k$, is exponentially generic in $\mathcal{C}_k$. Let G be given by a $C'(1/6)$-presentation on the generators $a_1, \dots, a_k$ where all the defining relations r have the property that every subword of r of length $|r|/6$ involves all the generators $a_1, \dots, a_k$. Then every proper subset of $a_1, \dots, a_k$ freely generates a subgroup of G. It now follows from part (1) of Corollary 3.4 that $\mathcal{P}$ is monotone low-density random. $\square$

One can regard property $\mathcal{G}$ from Corollary 5.4 above as a version of Magnus' Freiheitssatz for random groups. An asymptotic version of the Freiheitssatz using another model introduced by Gromov [8] was obtained by Cherix and Schaeffer [6].

Similar arguments to those used above yield:

Proposition 5.5. *Let $\mathcal{G} = (\mathcal{G}_k)_{k\ge 2}$ where $\mathcal{G}_k$ is the property that a finite group presentation on $a_1, \dots, a_k$ defines a group of rank k. Then $\mathcal{G}$ is monotone low-density random but not d-random for any $d > 0$.*

Proof. We have already observed in Theorem 4.2 that $\mathcal{G}$ is monotone low-density random. Let $\mathcal{G}' = (\mathcal{G}'_k)_{k\ge 2}$ where $\mathcal{G}'_k$ is the property that for a finite presentation on $a_1, \dots, a_k$ none of the defining relations are primitive in $F(a_1, \dots, a_k)$. Clearly, if $G = \langle a_1, \dots, a_k | r_1, \dots, r_m\rangle$ and some r_i is a primitive element in $F(a_1, \dots, a_k)$ (that is r_i belongs to some free basis of $F(a_1, \dots, a_k)$) then $rk(G) \le k - 1$. Thus $\mathcal{G}_k \subseteq \mathcal{G}'_k$ and $\mathcal{G} \subseteq \mathcal{G}'$. It suffices to show that $\mathcal{G}'$ is not d-random for any $d > 0$.

Let $\mathcal{P}_k \subseteq \mathcal{C}_k$ be the set of all non-primitive elements in $\mathcal{C}_k$. Note that for any freely reduced word $w \in F(a_2, \dots a_k)$ the element $a_1 w$ is primitive in $F(a_1, \dots, a_k)$. Hence

$$\gamma(n, \overline{\mathcal{P}_k}) \ge \gamma(n-1, F_{k-1}) = (2k-2)(2k-3)^{n-2}.$$

Therefore

$$1 \ge t(\mathcal{P}_k) \ge t'(\mathcal{P}_k) \ge \frac{\log(2k-3)}{\log(2k-1)} \to_{k\to\infty} 1.$$

Part (2) of Corollary 3.4 implies that $\mathcal{G}'$ is not d-random for any $d > 0$. $\square$

6. A bounded freeness property

In this section we will show that for every fixed integer $L \ge 2$ there is some $0 < d < 1$ such that the property of a finitely presented group that all its L-generated subgroups of infinite index are free is monotone d-random.

First, we need to investigate the genericity entropy of the set of non-(μ, L)-readable words.

Definition 6.1. Let $L \ge 2$ and $k \ge 2$ be integers. Let $0 < \mu < 1$. We say that a freely reduced word $v \in F(a_1, \dots, a_k)$ is (μ, L)-*readable* if there exists a finite connected graph Γ with the following properties:

(1) Every edge e of Γ is labelled by some element $s(e)$ of $\{a_1, \dots, a_k\}^{\pm 1}$ so that for every edge e we have $s(e^{-1}) = s(e)^{-1}$.
(2) The graph Γ is folded.
(3) The fundamental group of Γ is free of rank at most L.
(4) The graph Γ has at least one vertex of degree $< 2k$.
(5) The graph Γ has at most two degree-1 vertices.
(6) There exists an immersed path in Γ labelled v.

(7) The volume of Γ (that is, the number of non-oriented edges) is at most $\mu|v|$.

Let $\mathcal{Q}_k(\mu, L)$ be the set of all (μ, L)-readable words in $F(a_1, \dots, a_k)$.

Proposition 6.2. *Let $k > L$.*
We have

$$\gamma(n, \mathcal{Q}_k(\mu, L)) \le C(\mu n)^{3L+1}(6L)^n(2k-1)^{\mu n}.$$

where $C > 0$ is independent of n.

Proof. Recall that an *arc* in Γ is an immersed edge-path where every intermediate vertex of the path has degree two in Γ.

Note that if Γ is a finite connected graph with fundamental group free of rank $\le L < k$, then Γ necessarily has a vertex of degree $< 2k$. Thus condition (4) of Definition 6.1 is redundant in this case.

Let $L > k$ and $0 < \mu < 1$ be fixed. Let $v \in F(a_1, \dots, a_k)$ be a (μ, L)-readable word with $|v| = n$.

First, we estimate the number of labelled graphs Γ as in Definition 6.1 where v can be read.

There are $\le C_0 = C_0(L)$ topological types of the graphs Γ arising in the definition of a (μ, L)-readable word. Since $\pi_1(\Gamma)$ has rank at most L and Γ has at most two degree-1 vertices, it follows that Γ has $\le 3L$ non-directed maximal arcs and $\le 6L$ directed maximal arcs.

The sum of the length of theses arcs is $\le \mu n$. The number of ways to represent a positive integer N as a sum

$$N = N_1 + \cdots + N_{3L}$$

where N_i are non-negative integers is

$$\frac{(N+3L-1)!}{N!(3L-1)!} \le (N+3L-1)^{3L}.$$

Hence the number of ways to write a sum

$$N_1 + \cdots + N_{3L} \le \mu n$$

is $\le C_1(\mu n)^{3L+1}$, where C_1 is independent of n. For each decomposition $N_1 + \cdots + N_{3L} \le \mu n$ the number of ways to assign the maximal arcs of Γ labels $v_1, \dots v_{3L} \in F(a_1, \dots, a_k)$ with $|v_i| = N_i$ is

$$\le C_2(2k-1)^{\mu n}$$

where $C_2 > 0$ does not depend on n.

Thus there are at most $C_0C_1C_2(\mu n)^{3L+1}(2k-1)^{\mu n}$ relevant labelled graphs Γ as in Definition 6.1

For a fixed Γ, if v can be read in Γ then v is the label of a path

$$p_1', p_2, \dots, p_{s-1}, p_s'$$

where p_i are oriented maximal arcs, p_1', p_s' are oriented arcs and $s \le |v| = n$. By passing to a subgraph of Γ if necessary we may assume that p_1' and p_2' are maximal arcs as well. Thus v is the label of a path $\alpha = p_1, p_2, \dots, p_{s-1}, p_s$ where p_i are directed maximal arcs in Γ and where $s \le n = |v|$. Since $s \le n$ and Γ has $\le 6L$ oriented maximal arcs, there are $\le (6L)^n$ combinatorial possibilities to express α as a word in the alphabet of $6L$ letters corresponding to the directed maximal arcs.

Hence the total number of possibilities for v is

$$\gamma(n, \mathcal{Q}_k(\mu, L)) \le C_0 C_1 C_2 (\mu n)^{3L+1} (6L)^n (2k-1)^{\mu n},$$

as required.

□

Definition 6.3. Let $k \ge$, $L \ge 2$ be integers and let $0 < \mu < 1$. We say that a cyclically reduced word $w \in F(a_1, \dots, a_k)$ is (μ, L)-*good* if no cyclic permutation of $w^{\pm 1}$ contains a subword v of length $\ge |w|/2$ such that v is (μ, L)-readable.

Lemma 6.4. *Let $k > L \ge 2$ and let $0 < \mu < 1$. Let $\mathcal{Y}_k = \mathcal{Y}_k(\mu, L) \subseteq \mathcal{C}_k$ be the set of all cyclically reduced (μ, L)-good words. Then*

$$t(\mathcal{Y}_k) \le \frac{((\mu+1)/2)\log(2k-1) + (1/2)\log(6L)}{\log(2k-1)}.$$

Proof. Let $w \in \overline{\mathcal{Y}_k}$ with $n = |w|$. There are at most $2n$ cyclic permutations of $w^{\pm 1}$ and at least one of them has an initial segment v of length $n/2$ such that v is (μ, L)-readable. Hence by Proposition 6.2 the number of possibilities for w is

$$\gamma(n, \overline{\mathcal{Y}_k}) \le A(2n)(\mu n/2)^{3L+1}(6L)^{n/2}(2k-1)^{\mu n/2}(2k-1)^{n/2},$$

where $A > 0$ is independent of n. Hence

$$t(\mathcal{Y}_k) = \limsup_{n\to\infty} \frac{\log \gamma(n, \overline{\mathcal{Y}_k})}{n \log(2k-1)} \le$$

$$\limsup_{n\to\infty} \frac{(\frac{n}{2} + \frac{\mu n}{2})\log(2k-1) + \frac{n}{2}\log 6L + \log(2An) + (3L+1)\log\frac{\mu n}{2}}{n\log(2k-1)} =$$

$$= \frac{(\frac{\mu+1}{2})\log(2k-1) + \frac{1}{2}\log(6L)}{\log(2k-1)}.$$

□

The results of Section 4 of [1] imply:

Proposition 6.5. *Let $L, k \ge 2$ be integers. Let $0 < \mu < 1$ and $0 < \lambda < 1$ be such that*

$$0 < \lambda \le \frac{\mu}{15L + 3\mu} \le \frac{1}{6}.$$

Let $G = \langle a_1, \dots, a_k | r_1, \dots, r_m \rangle$ be such that

(1) *The above presentation of G satisfies the small cancellation condition $C'(\lambda)$.*

(2) *All* $r_1, \dots, r_m$ *are cyclically reduced words that are not proper powers in* $F(a_1, \dots, a_k)$.
(3) *Each* r_i *is* (μ, L)*-good.*

Then every L*-generated subgroup of infinite index in* G *is free.*

Theorem 6.6. *For every integer* $L \geq 2$ *there is some* $d > 0$ *such that the property of finitely presented groups for all* L*-generated subgroups of infinite index to be free is monotone* d*-random.*

Proof. Let $L \geq 2$ be a fixed integer.

It is well-known and easy to see that conditions (1) and (2) from Proposition 6.5 are monotone random [9]. Thus it suffices to deal with condition (3) of Proposition 6.5.

Choose $0 < \lambda, \mu < 1$ so that

$$0 < \lambda \leq \frac{\mu}{15L + 3\mu} \leq \frac{1}{6}.$$

We have

$$\lim_{k\to\infty} \frac{((\mu+1)/2)\log(2k-1) + (1/2)\log(6L)}{\log(2k-1)} = \frac{\mu+1}{2} < 1.$$

Choose ν so that $(\mu + 1)/2 < \nu < 1$. There exists an integer $k_0 > L$ such that for any $k \geq k_0$

$$\frac{((\mu+1)/2)\log(2k-1) + (1/2)\log(6L)}{\log(2k-1)} \leq \nu < 1.$$

Thus by Lemma 6.4 for $k \geq k_0$ we have

$$t(\mathcal{Y}_k) \leq \nu < 1.$$

Recall that by Theorem 4.2 the property of having all L-generated subgroups of infinite index being free is monotone low-density random with density sequence $(d(k))_{k\geq 2}$. Put $d_0 := \min\{d(2), \dots, d(k_0 - 1), 1 - \nu\}$. Then by Proposition 3.3 the property of having all L-generated subgroups of infinite index being free is monotone d-random for any $0 < d < d_0$. □

Remark 6.7. In [1] Arzhantseva gave a proof of exponential genericity in $F(a_1, \dots, a_k)$ of non-(μ, L)-readable words, assuming that μ is small enough. However, the estimates on the growth of (μ, L)-readable words obtained there are insufficient for our purposes in the proof of Theorem 6.6. Let $\mathcal{P}_k \subseteq \mathcal{C}_k$ be the set of all non-(μ_k, L)-readable cyclically reduced words in $F(a_1, \dots, a_k)$, where $0 < \mu_k < 1$ satisfies

$$0 < \mu_k < \frac{1}{3L}\log_{2k}\left(1 + \frac{1}{2(2k-1)^{3L} - 2}\right).$$

A crucial estimate in Lemma 3 of [1] shows that

$$(*) \qquad \gamma(n, \overline{\mathcal{P}_k}) \leq A\left((2k-1)^{3L} - \frac{1}{2}\right)^{n/3L}.$$

This yields

$$t(\mathcal{P}_k) \leq \frac{\log\left((2k-1)^{3L} - \frac{1}{2}\right)}{3L\log(2k-1)} \to_{k\to\infty} 1,$$

where convergence to 1 in the last limit is easily seen by applying l'Hôpital's rule. Therefore we needed an estimate different from $(*)$ for the number of (μ, L)-readable words in Proposition 6.2. That estimate allowed us to obtain bounds on the genericity entropy of the set of (μ, L)-good words that are independent of k for sufficiently large k. On the other hand, we still needed the results of [1] obtained via the estimate $(*)$ to deal with the case of "small" k with $k < k_0$ in the proof of Theorem 6.6.

References

[1] G. Arzhantseva, *On groups in which subgroups with a fixed number of generators are free,*(Russian) Fundam. Prikl. Mat. **3** (1997), no. 3, 675–683.

[2] G. Arzhantseva, *Generic properties of finitely presented groups and Howson's theorem,* Comm. Algebra **26** (1998), 3783–3792.

[3] G. Arzhantseva, *A property of subgroups of infinite index in a free group,* Proc. Amer. Math. Soc. **128** (2000), 3205–3210.

[4] G. Arzhantseva, and P.-A. Cherix, *On the Cayley graph of a generic finitely presented group.* Bull. Belg. Math. Soc. Simon Stevin **11** (2004), no. 4, 589–601

[5] G. Arzhantseva and A. Ol'shanskii, *Genericity of the class of groups in which subgroups with a lesser number of generators are free,* (Russian) Mat. Zametki **59** (1996), no. 4, 489–496

[6] P.-A. Cherix and G. Schaeffer, *An asymptotic Freiheitssatz for finitely generated groups,* Enseign. Math. (2) **44** (1998), 9–22.

[7] E. Ghys, *Groupes aléatoires (d'après Misha Gromov,...),* Astérisque **294** (2004), 173–204

[8] M. Gromov, *Hyperbolic Groups*, in "Essays in Group Theory (G.M.Gersten, editor)", MSRI publ. **8**, 1987, 75–263

[9] M. Gromov, *Random walks in random groups*, Geom. Funct. Analysis **13** (2003), no. 1, 73–146

[10] V. Kaimanovich, I. Kapovich and P. Schupp, *The Subadditive Ergodic Theorem and generic stretching factors for free group automorphisms,* Israel J. Math. **157** (2007), 1–46

[11] I. Kapovich and P. Schupp, *Genericity, the Arzhantseva-Ol'shanskii method and the isomorphism problem for one-relator groups*, Math. Annalen **331** (2005), no. 1, 1–19

[12] I. Kapovich, P. Schupp, and V. Shpilrain, *Generic properties of Whitehead's algorithm and isomorphism rigidity of random one-relator groups*, Pacific J. Math. **223** (2006), no. 1, 113–140

[13] I. Kapovich and P. Schupp, *Delzant's T-ivariant, one-relator groups and Kolmogorov complexity*, Commentari Math. Helv. **80** (2005), no. 4, 911–933

[14] I. Kapovich and P. Schupp, *Random quotients of the modular group are rigid and essentially incompressible*, preprint, 2006; http://arxiv.org/abs/math.GR/0604343

[15] Y. Ollivier, *Critical densities for random quotients of hyperbolic groups.* C. R. Math. Acad. Sci. Paris **336** (2003), no. 5, 391–394

[16] Y. Ollivier, *Sharp phase transition theorems for hyperbolicity of random groups.* Geom. Funct. Anal. **14** (2004), no. 3, 595–679

[17] Y. Ollivier, *Cogrowth and spectral gap of generic groups.* Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 289–31

[18] Y. Ollivier, *Effondrement de quotients aleatoires de groupes hyperboliques avec torsion.* C. R. Math. Acad. Sci. Paris **341** (2005), no. 3, 137–140
[19] I. Rivin, *Growth in free groups (and other stories)*, preprint, 1999; http://www.arxiv.org/math.CO/9911076
[20] A. Zuk, *Property (T) and Kazhdan constants for discrete groups.* Geom. Funct. Anal. **13** (2003), no. 3, 643–670.

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