

# ON THE ABSENCE OF MCSHANE-TYPE IDENTITIES FOR THE OUTER SPACE

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ABSTRACT. A remarkable result of McShane states that for a punctured torus with a complete finite volume hyperbolic metric we have

$$\sum_{\gamma} \frac{1}{e^{\ell(\gamma)} + 1} = \frac{1}{2}$$

where  $\gamma$  varies over the homotopy classes of essential simple closed curves and  $\ell(\gamma)$  is the length of the geodesic representative of  $\gamma$ .

We prove that there is no reasonable analogue of McShane's identity for the Culler-Vogtmann outer space of a free group.

## 1. INTRODUCTION

Let  $T$  be the one-punctured torus and let  $\rho$  be a complete finite-volume hyperbolic structure on  $T$ . Let  $\mathcal{S}$  be the set of all free homotopy classes of essential simple closed curves in  $T$  that are not homotopic to the puncture. Denote

$$E(\rho) := \sum_{\gamma \in \mathcal{S}} \frac{1}{e^{\ell_{\rho}(\gamma)} + 1},$$

where  $\ell_{\rho}(\gamma)$  is the smallest  $\rho$ -length among all curves representing  $\gamma$ . Thus  $E$  can be regarded as a function on the Teichmüller space of  $T$ . A remarkable result of McShane [9] shows that this function is constant and that

$$(*) \quad E(\rho) = \frac{1}{2}$$

for every  $\rho$ . We refer to  $(*)$  as *McShane's identity* for  $T$ . Since the thesis of McShane [9], other proofs of McShane's identity for the punctured torus have been produced (particularly, see the work of Bowditch [3]) and McShane's identity has been generalized to other hyperbolic surfaces and other contexts [4, 10, 1, 2, 13, 14]. Note that if  $\psi$  is an element of the mapping class group of  $T$  then  $\psi$  permutes the elements of  $\mathcal{S}$  and hence, clearly,  $E(\rho) = E(\psi\rho)$ . Thus  $E$  obviously factors through to a function on the moduli space of  $T$  and  $(*)$  says that this function is identically equal to  $1/2$ .

Let  $F_k = F(a_1, \dots, a_k)$  be a free group of rank  $k \geq 2$  with a free basis  $A = \{a_1, \dots, a_k\}$ . For  $F_k$  the best analogue of the Teichmüller space is the so-called Culler-Vogtmann *outer space*  $CV(F_k)$ . Instead of actions on the hyperbolic plane the elements of the outer space are represented by minimal discrete isometric actions of  $F_k$  on  $\mathbb{R}$ -trees. Equivalently, one can think about a point of the outer space as being represented by a *marked volume-one metric graph structure* on  $F_k$ , that is,

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an isomorphism  $\phi : F_k \rightarrow \pi_1(\Gamma, p)$ , where  $\Gamma$  is a finite graph without degree-one and degree-two vertices, equipped with a *metric structure*  $\mathcal{L}$  that assigns to each non-oriented edge of  $\Gamma$  a positive number called the *length* of this edge. The *volume* of a metric structure on  $\Gamma$  is the sum of the lengths of all non-oriented edges of  $\Gamma$ . As we noted, the metric structures that appear in the description of the points of the outer space, given above, are required to have volume equal to one. If  $(\phi : F_k \rightarrow \pi_1(\Gamma, p), \mathcal{L})$  represents a point of the outer space, the metric structure  $\mathcal{L}$  naturally lifts to the universal cover  $\tilde{\Gamma}$ , turning  $\tilde{\Gamma}$  into an  $\mathbb{R}$ -tree  $X$ . The group  $F_k$  acts on this  $\mathbb{R}$ -tree  $X$  via  $\phi$  by isometries minimally and discretely with the quotient being equal to  $\Gamma$ . Similarly to the marked length spectrum in the Teichmüller space context, a marked metric graph structure  $(\phi : F_k \rightarrow \pi_1(\Gamma, p), \mathcal{L})$  defines a *hyperbolic length function*  $\ell : \mathcal{C}_k \rightarrow \mathbb{R}$  where  $\mathcal{C}_k$  is the set of all nontrivial conjugacy classes in  $F_k$ . If  $g \in F_k$ , then  $\ell([g])$  is the translation length of  $g$  considered as the isometry of the  $\mathbb{R}$ -tree  $X$  described above. Alternatively, we can think about  $\ell([g])$  as follows:  $\ell([g])$  is the  $\mathcal{L}$ -length of the shortest free homotopy representative of the curve  $\phi(g)$  in  $\Gamma$ , that is, the  $\mathcal{L}$ -length of the "cyclically reduced" form of  $\phi(g)$  in  $\Gamma$ . Two volume-one metric graph structures on  $F_k$  represent the same point of  $CV(F_k)$  if and only if their corresponding hyperbolic length functions are equal, or, equivalently, if the corresponding  $\mathbb{R}$ -trees are  $F_k$ -equivariantly isometric.

It is natural to ask if there is an analogue of McShane's identity in the outer space context. The (right) action of  $\psi \in Out(F_k)$  on  $CV(F_k)$  takes a hyperbolic length function  $\ell$  to  $\ell \circ \psi$ , that is,  $\psi$  simply permutes the domain  $\mathcal{C}_k$  of  $\ell$ . Therefore the real question, as in the Teichmüller space case, is if there is an analogue of McShane's identity for the *moduli space*  $\mathcal{M}_k = CV(F_k)/Out(F_k)$ . The elements of  $\mathcal{M}_k$  are represented by unmarked finite connected volume-one metric graphs  $(\Gamma, \mathcal{L})$  without degree-one and degree-two vertices and with  $\pi_1(\Gamma) \simeq F_k$ .

To simplify the picture, and also since our results will be negative, we will consider a subset  $\Delta_k$  of  $CV(F_k)$  consisting of all volume-one metric structures on the wedge  $W_k$  of  $k$  circles wedged at a base-vertex  $v_0$ . We orient the circles and label them by  $a_1, \dots, a_k$ . This gives us an identification  $\pi_1(W_k, v_0) = F(a_1, \dots, a_k)$  of  $F_k = F(a_1, \dots, a_k)$  with  $\pi_1(W_k, v_0)$ , so that indeed  $\Delta_k \subseteq CV(F_k)$ .

A volume-one metric structure  $\mathcal{L}$  on  $W_k$  is a  $k$ -tuple  $(\mathcal{L}(a_1), \dots, \mathcal{L}(a_k))$  of positive numbers with  $\sum_{i=1}^k \mathcal{L}(a_i) = 1$ . Thus  $\Delta_k$  has the natural structure of an open  $(k-1)$ -dimensional simplex in  $\mathbb{R}^k$ . As in the general outer space context, every  $\mathcal{L} \in \Delta_k$  defines a hyperbolic length-function  $\ell_{\mathcal{L}} : \mathcal{C}_k \rightarrow \mathbb{R}$ , where for  $g \in F_k$   $\ell_{\mathcal{L}}([g])$  is the  $\mathcal{L}$ -length of the cyclically reduced form of  $g$  in  $F_k = F(a_1, \dots, a_k)$ . The open simplex  $\Delta_k$  has a distinguished point  $\mathcal{L}_* := (\frac{1}{k}, \dots, \frac{1}{k})$ . Note that for every  $[g] \in \mathcal{C}_k$  we have  $\ell_{\mathcal{L}_*}([g]) = \|g\|/k$ , where  $\|g\|$  is the cyclically reduced length of  $g$  in  $F_k = F(a_1, \dots, a_k)$ .

There is no perfect analogue for the notion of a simple closed curve in the free group context. The closest such analogue is given by *primitive elements*, that is, elements belonging to some free basis of  $F_k$ . Let  $\mathcal{P}_k$  be denote the set of conjugacy classes of primitive elements of  $F_k$ . We will consider two versions of possible generalizations of McShane's identity for free groups: the first involving all conjugacy classes in  $F_k$  and the second involving the conjugacy classes of primitive elements of  $F_k$ . We will see that, under some reasonable assumptions, there are no analogues of McShane's identity in either context.

**Definition 1.1** (McShane-type functions on  $\Delta_k$ ). Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a monotone non-increasing function and let  $k \geq 2$ . Define

$$C_f : \Delta_k \rightarrow (0, \infty], \quad C_f(\mathcal{L}) = \sum_{w \in \mathcal{C}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k$$

and

$$P_f : \Delta_k \rightarrow (0, \infty], \quad P_f(\mathcal{L}) = \sum_{w \in \mathcal{P}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k.$$

Obviously,  $0 < P_f < C_f \leq \infty$  on  $\Delta_k$ .

Motivated by McShane's identity, it is interesting to ask if there exist functions  $f$  such that either  $C_f$  or  $P_f$  is constant on  $\Delta_k$ . To make the question meaningful we need to require  $P_f$  (or, correspondingly,  $C_f$ ) be finite at some point  $\mathcal{L} \in \Delta_k$ . Thus it is necessary to assume that  $\lim_{x \rightarrow \infty} f(x) = 0$  and that this convergence to zero is sufficiently fast.

We establish the following negative results regarding the existence of analogues of McShane's identity in the outer space context:

**Theorem A.** *Let  $k \geq 2$  be an integer and let  $F = F(a_1, \dots, a_k)$ . Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a monotone non-increasing function such that:*

(1)

$$\limsup_{x \rightarrow \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

(2)

$$\liminf_{x \rightarrow \infty} f(x)^{1/x} > 0.$$

Then:

- (a) *We have  $P_f \leq C_g < \infty$  on some neighborhood  $U$  on  $\mathcal{L}_*$  in  $\Delta_k$  (moreover, only the assumption (1) on  $f$  above is required for this conclusion).*
- (b) *We have  $C_f \neq \text{const}$  on  $\Delta_k$ .*
- (c) *If  $k \geq 3$  then  $P_f \neq \text{const}$  on  $\Delta_k$ .*

The assumptions on  $f(x)$  in Theorem A require  $f(x)$  to decay both at least and at most exponentially fast; condition (1) assures that the value of  $C_f$  is finite near  $\mathcal{L}_*$ . The idea of the proof of parts (b) and (c) of Theorem A uses the notion of *volume entropy* for a metric structure  $\mathcal{L}$  on  $W_k$  (see [6, 11, 8]). Roughly speaking, there are points  $\mathcal{L}$  near the the boundary of  $\Delta_k$  where the exponential growth rate, as  $R \rightarrow \infty$ , of the number of conjugacy classes with  $\ell_{\mathcal{L}}$ -length at most  $R$  is bigger than the exponential rate of decay of the function  $f$ . This forces  $C_f$  to be equal to  $\infty$  at  $\mathcal{L}$ .

For  $k = 2$  the set of conjugacy classes of primitive elements has quadratic rather than exponential growth. Therefore we modify the assumptions on  $f(x)$  accordingly and obtain a somewhat stronger conclusion than in part (c) of Theorem A. For  $k = 2$  the open 1-dimensional simplex  $\Delta_2 \subseteq \mathbb{R}^2$  consists of all pairs  $\mathcal{L}_t := (t, 1-t)$  where  $t \in (0, 1)$ . Therefore we may identify  $\Delta_2$  with  $(0, 1)$  and define  $P_f(t) := P_f(\mathcal{L}_t)$ . With this convention we prove:

**Theorem B.** *Let  $k = 2$  and  $F = F(a, b)$ . Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a monotone non-increasing function such that:*

- (1) *We have  $f''(x) > 0$  for every  $x > 0$ .*
- (2) *There is some  $\epsilon > 0$  such that  $\lim_{x \rightarrow \infty} x^{3+\epsilon} f(x) = 0$ .*

Then the following hold:

- (a) We have  $0 < P_f(t) < \infty$  for every  $t \in (0, 1)$ .
- (b) The function  $P_f(t)$  is strictly convex on  $(0, 1)$  and achieves a unique minimum at  $t_0 = 1/2$ . In particular,  $P_f(t)$  is not a constant locally near  $t_0 = 1/2$  and thus  $P_f \neq \text{const}$  on  $(0, 1)$ .

The proof of Theorem B uses convexity considerations as well as some results about the explicit structure of primitive elements in  $F(a, b)$  [5, 12].

Finally, we combine the volume entropy and the convexity ideas to obtain:

**Theorem C.** *Let  $k \geq 2$  and let  $f : (0, \infty) \rightarrow (0, \infty)$  be a monotone decreasing function such that the following hold:*

- (1) *The function  $f(x)$  is strictly convex on  $(0, \infty)$ .*
- (2)

$$\limsup_{x \rightarrow \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

*Then there exists a convex neighborhood  $U$  of  $\mathcal{L}_*$  in  $\Delta_k$  such that  $0 < P_f < C_f < \infty$  on  $U$  and both  $C_f$  and  $P_f$  are strictly convex on  $U$ . In particular,  $C_f \neq \text{const}$  on  $U$  and  $P_f \neq \text{const}$  on  $U$ .*

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## 2. VOLUME ENTROPY

In this section we will prove Theorem A, which is obtained as a combination of Theorem 2.4 and Theorem 2.5 below.

**Convention 2.1.** For the remainder of this section let  $k \geq 2$  be an integer and  $F_k = F(a_1, \dots, a_k)$  be free of rank  $k$  with a free basis  $A = \{a_1, \dots, a_k\}$ . We identify  $F_k$  with  $\pi_1(W_k, v_0)$ , as explained in the introduction. For  $g \in F_k$  we denote by  $|g|$  the freely reduced length of  $g$  with respect to  $A$  and we denote by  $\|g\|$  the cyclically reduced length of  $g$  with respect to  $A$ .

We denote by  $CR_k$  the set of all cyclically reduced elements of  $F_k$  with respect to  $A$ .

Let  $\mathcal{L}$  be a metric graph structure on  $W_k$ . For every  $g \in F_k$  there is a unique edge-path in  $W_k$  labelled by the freely reduced form of  $g$  with respect to  $A$ . We denote the  $\mathcal{L}$ -length of that path by  $\mathcal{L}(g)$ . As before, we denote by  $\ell_{\mathcal{L}} : \mathcal{C}_k \rightarrow \mathbb{R}$  the hyperbolic length function corresponding to  $\mathcal{L}$ . Thus if  $g \in F_k$  then  $\ell_{\mathcal{L}}([g]) = \mathcal{L}(u)$  where  $u$  is the cyclically reduced form of  $g$  with respect to  $A$ .

**Definition 2.2** (Volume Entropy). Let  $\mathcal{L}$  be a metric structure on  $W_k$ . The *volume entropy*  $h_{\mathcal{L}}$  of  $\mathcal{L}$  is defined as

$$h_{\mathcal{L}} = \lim_{R \rightarrow \infty} \frac{\log \#\{g \in F_k : \mathcal{L}(g) \leq R\}}{R}.$$

It is well-known and easy to see that the limit in the above expression exists and is finite. We refer the reader to [6, 11, 8] for a detailed discussion of volume entropy in the context of metric graphs.

**Proposition 2.3.** *Let  $k \geq 2$  and  $\mathcal{L}$  be as in definition 2.2.*

*Then the limits*

$$h'_{\mathcal{L}} = \lim_{R \rightarrow \infty} \frac{\log \#\{g \in CR_k : \mathcal{L}(g) \leq R\}}{R}.$$

and

$$h''_{\mathcal{L}} = \lim_{R \rightarrow \infty} \frac{\log \#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \leq R\}}{R}.$$

exist and

$$h_{\mathcal{L}} = h'_{\mathcal{L}} = h''_{\mathcal{L}}.$$

*Proof.* Let  $M := \max\{|a_i|_{\mathcal{L}} : i = 1, \dots, k\}$  and  $m := \min\{|a_i|_{\mathcal{L}} : i = 1, \dots, k\}$

For each  $g \in F$  there exists a cyclically reduced word  $v_g$  such that  $|g| = |v_g|$  and such that  $g$  and  $v_g$  agree except possibly in the last letter. Then  $|\mathcal{L}(g) - \mathcal{L}(v_g)| \leq M$ . Moreover, the function  $F_k \rightarrow CR_k, g \mapsto v_g$  is at most  $2k$ -to-one. Therefore for every integer  $R > 0$

$$\#\{g \in CR_k : \mathcal{L}(g) \leq R\} \leq \#\{g \in F_k : \mathcal{L}(g) \leq R\} \leq 2k\#\{g \in CR_k : \mathcal{L}(g) \leq R+M\}$$

and

$$\#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \leq R\} \leq \#\{g \in CR_k : \mathcal{L}(g) \leq R\} \leq \frac{R}{m} \#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \leq R\}.$$

This implies the statement of the proposition.  $\square$

**Theorem 2.4.** *Let  $k \geq 2$  be an integer and let  $F = F(a_1, \dots, a_k)$ . Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a monotone non-increasing function such that:*

(1)

$$\limsup_{x \rightarrow \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

(2)

$$\liminf_{x \rightarrow \infty} f(x)^{1/x} > 0.$$

Then:

(a) *We have  $0 < C_f < \infty$  on some neighborhood  $U$  on  $\mathcal{L}_*$  in  $\Delta_k$  (moreover, only the assumption (1) on  $f$  above is required for this conclusion).*

(b) *We have  $C_f \neq \text{const}$  on  $\Delta_k$ .*

*Proof.* The assumptions on  $f(x)$  imply that there exist  $N > 0$  and  $0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^k}$  such that for every  $x \geq N$

$$\sigma_1^x \leq f(x) \leq \sigma_2^x.$$

Let  $\mathcal{L}_* = (\frac{1}{k}, \dots, \frac{1}{k}) \in \Delta_k$ . For any  $g \in F_k$  we have  $\mathcal{L}_*(g) = |g|/k$ . Then an easy direct computation shows that  $h_{\mathcal{L}_*} = k \log(2k-1)$ , so that  $e^{h_{\mathcal{L}_*}} = (2k-1)^k < \frac{1}{\sigma_2}$ . Since the volume entropy  $h$  is a continuous function on  $\Delta_k$  (see, for example, [6]), there exist a neighborhood  $U$  of  $\mathcal{L}_*$  in  $\Delta_k$  and  $0 < c < \frac{1}{\sigma_2}$  such that  $e^{h_{\mathcal{L}}} < c$  for every  $\mathcal{L} \in U$ .

Observe now that  $C_f < \infty$  on  $U$ . Let  $\mathcal{L} \in U$  be arbitrary. There there exist  $M > 0$  and  $c_1$  with  $c < c_1 < \frac{1}{\sigma_2}$  such that for every integer  $R > 0$  we have

$$\#\{w \in \mathcal{C}_k : \ell_{\mathcal{L}}(w) \leq R\} \leq M c_1^R$$

Therefore

$$\begin{aligned} C_f(\mathcal{L}_*) &= \sum_{w \in \mathcal{C}_k} f(\ell_{\mathcal{L}}(w)) = \sum_{i=0}^{\infty} \sum_{w \in \mathcal{C}_k, i < \ell_{\mathcal{L}}(w) \leq i+1} f(\ell_{\mathcal{L}}(w)) \leq \\ &\sum_{i=0}^{\infty} \sum_{w \in \mathcal{C}_k, i < \ell_{\mathcal{L}}(w) \leq i+1} f(i) \leq \sum_{i=0}^{\infty} M c_1^{i+1} f(i) < \infty \end{aligned}$$

where the last inequality holds since  $c_1 < \frac{1}{\sigma_2}$  and  $f(x) \leq \sigma_2^x$  for all  $x \geq N$ . Thus indeed  $C_f(\mathcal{L}) < \infty$ , so that  $C_f < \infty$  on  $U$ .

For  $0 < t < \frac{1}{k-1}$  let  $\mathcal{L}_t = (t, t, \dots, t, 1 - (k-1)t) \in \Delta_k$ . Then, as follows from the proof of Theorem B of [6] (specifically the proof of Theorem 9.4 on page 25 of [6]),

$$\lim_{t \rightarrow 0} h_{\mathcal{L}_t} = \infty.$$

Indeed, let  $\Gamma$  be the subgraph of  $W_k$  consisting of the loops labelled by  $a_1$  and  $a_k$ . The restriction  $\mathcal{L}'_t$  of  $\mathcal{L}_t$  to  $\Gamma$  is a metric structure on  $\Gamma$  of volume  $1 - (k-2)t$ . Therefore  $\frac{1}{1-(k-2)t} \mathcal{L}'_t$  is a volume-one metric structure on  $\Gamma$  with respect to which the length of  $a_1$  goes to 0 as  $t \rightarrow 0$ . Therefore, as established in the proof of Theorem 9.4 of [6],

$$\lim_{t \rightarrow 0} h_{\frac{1}{1-(k-2)t} \mathcal{L}'_t} = \infty.$$

However,

$$h_{\mathcal{L}'_t} = \frac{1}{1 - (k-2)t} h_{\frac{1}{1-(k-2)t} \mathcal{L}'_t}$$

and therefore

$$\lim_{t \rightarrow 0} h_{\mathcal{L}'_t} = \infty.$$

It is obvious from the definition of volume entropy that  $h_{\mathcal{L}'_t} \leq h_{\mathcal{L}_t}$  and hence

$$\lim_{t \rightarrow 0} h_{\mathcal{L}_t} = \infty,$$

as claimed.

Hence there exists  $t_0 \in (0, \frac{1}{k-1})$  such that for every  $t \in (0, t_0)$  we have  $e^{h_{\mathcal{L}_t}} > \frac{1}{\sigma_1} + 2$ . Let  $t \in (0, t_0)$  be arbitrary. We claim that  $C_f(\mathcal{L}_t) = \infty$ .

Since  $e^{h_{\mathcal{L}_t}} > \frac{1}{\sigma_1} + 2$ , by Proposition 2.3 there is  $R_0 > N > 0$  such that for every  $R \geq R_0$  we have

$$\#\{w \in \mathcal{C}_k : \mathcal{L}_t(w) \leq R\} \geq \left(\frac{1}{\sigma_1} + 1\right)^R.$$

For every  $R \geq R_0$

$$\begin{aligned} C_f(\mathcal{L}_t) &= \sum_{w \in \mathcal{C}_k} f(\mathcal{L}_t(w)) \geq \sum_{w \in \mathcal{C}_k, \mathcal{L}_t(w) \leq R} f(\mathcal{L}_t(w)) \geq \\ &\sum_{w \in \mathcal{C}_k, \mathcal{L}_t(w) \leq R} f(R) \geq \left(\frac{1}{\sigma_1} + 1\right)^R f(R) \geq \left(\frac{1}{\sigma_1} + 1\right)^R \sigma_1^R = \\ &= (1 + \sigma_1)^R. \end{aligned}$$

Since this is true for every  $R \geq R_0$ , it follows that  $C_f(\mathcal{L}_t) = \infty$ .

Thus  $C_f(\mathcal{L}_*) < \infty$  while  $C_f(\mathcal{L}_t) = \infty$  for all sufficiently small  $t > 0$ . Therefore  $C_f \neq \text{const}$  on  $\Delta_k$ .  $\square$

**Theorem 2.5.** *Let  $k \geq 3$  be an integer and let  $F = F(a_1, \dots, a_k)$ . Let  $f : (0, \infty) \rightarrow (0, \infty)$  be as in Theorem 2.4.*

*Then:*

- (a) *We have  $P_f \leq C_f < \infty$  on some neighborhood  $U$  on  $\mathcal{L}_*$  in  $\Delta_k$ .*
- (b) *We have  $P_f \neq \text{const}$  on  $\Delta_k$ .*

*Proof.* Again, by assumptions on  $f(x)$ , there exist there exist  $N > 0$  and  $0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^k}$  such that for every  $x \geq N$

$$\sigma_1^x \leq f(x) \leq \sigma_2^x.$$

By Definition  $0 \leq P_f \leq C_f$ . By Theorem 2.4 we have  $C_f < \infty$  on some neighborhood  $U$  on  $\mathcal{L}_*$  in  $\Delta_k$  and hence  $P_f \leq C_f < \infty$  on  $U$ .

Put  $F_{k-1} := F(a_1, \dots, a_{k-1})$  so that  $F_k = F_{k-1} * \langle a_k \rangle$ . For  $0 < t < \frac{1}{k-2}$  let

$$\mathcal{L}_t := \left( \frac{t}{2}, \frac{t}{2}, \dots, \frac{t}{2}, \frac{1}{2} - (k-2)\frac{t}{2}, \frac{1}{2} \right) \in \Delta_k$$

and

$$\widehat{\mathcal{L}}_t := \left( \frac{t}{2}, \frac{t}{2}, \dots, \frac{t}{2}, \frac{1}{2} - (k-2)\frac{t}{2} \right) \in \frac{1}{2}\Delta_{k-1}.$$

Thus  $2\widehat{\mathcal{L}}_t \in \Delta_{k-1}$  is a volume-one metric structure on  $W_{k-1}$ . Since  $k \geq 3$ , we have  $k-1 \geq 2$  and hence, exactly as in the proof of Theorem 2.4,  $\lim_{t \rightarrow 0} h_{2\widehat{\mathcal{L}}_t} = \infty$ .

For  $R \geq 1$

$$b_{R,t} := \#\{g \in F_{k-1} : \widehat{\mathcal{L}}_t(g) \leq R\}.$$

Then

$$s_t := \lim_{R \rightarrow \infty} \frac{\log b_{R,t}}{R} = h_{\widehat{\mathcal{L}}_t} = 2h_{2\widehat{\mathcal{L}}_t}.$$

and therefore

$$\lim_{t \rightarrow 0} s_t = \infty.$$

Hence there exists  $0 < t_0 < \frac{1}{k-2}$  such that for every  $t \in (0, t_0)$  we have  $e^{s_t} > \frac{1}{\sigma_1} + 2$ .

Fix an arbitrary  $t \in (0, t_0)$ . Since  $e^{s_t} > \frac{1}{\sigma_1} + 2$ , by there is  $R_0 > N > 0$  such that for every  $R \geq R_0$  we have

$$b_{R,t} = \#\{g \in F_{k-1} : \widehat{\mathcal{L}}_t(g) \leq R\} \geq \left(\frac{1}{\sigma_1} + 1\right)^R.$$

Note that for every  $g \in F_{k-1}$  the element  $ga_k \in F_k$  is primitive in  $F$ . Moreover, if  $g_1 \neq g_2$  are distinct elements of  $F_{k-1}$  then  $g_1 a_k$  and  $g_2 a_k$  are not conjugate in  $F_k$ . Recall that by definition of  $\mathcal{L}_t$  we have  $\mathcal{L}_t(a_k) = \frac{1}{2}$ . For  $R \geq 1$  denote

$$p_{R,t} := \#\{w \in \mathcal{P}_k : \ell_{\mathcal{L}_t}(w) \leq R\}.$$

Then for every  $R \geq R_0 + \frac{1}{2}$  we have

$$p_{R,t} \geq b_{R-\frac{1}{2},t} \geq \left(\frac{1}{\sigma_1} + 1\right)^{R-\frac{1}{2}}.$$

Hence for every  $R \geq R_0 + \frac{1}{2}$

$$\begin{aligned} P_f(\mathcal{L}_t) &= \sum_{w \in \mathcal{P}_k} f(\mathcal{L}_t(w)) \geq \sum_{w \in \mathcal{P}_k, \ell_{\mathcal{L}_t}(w) \leq R} f(\mathcal{L}_t(w)) \geq \\ &\sum_{w \in \mathcal{P}_k, \ell_{\mathcal{L}_t}(w) \leq R} f(R) \geq \left(\frac{1}{\sigma_1} + 1\right)^{R-\frac{1}{2}} f(R) \geq \left(\frac{1}{\sigma_1} + 1\right)^{R-\frac{1}{2}} \sigma_1^R = \\ &= (1 + \sigma_1)^R \left(\frac{1}{\sigma_1} + 1\right)^{-\frac{1}{2}}. \end{aligned}$$

Since this is true for every  $R \geq R_0 + \frac{1}{2}$ , it follows that  $P_f(\mathcal{L}_t) = \infty$ .

Thus  $P_f(\mathcal{L}_*) < \infty$  while  $P_f(\mathcal{L}_t) = \infty$  for all sufficiently small  $t > 0$ . Therefore  $P_f \neq \text{const}$  on  $\Delta_k$ . □

### 3. PRIMITIVE ELEMENTS IN $F(a, b)$

In this section we will prove Theorem B.

**Convention 3.1.** Throughout this section let  $F_2 = F(a, b)$  be a free group of rank two.

Let  $\alpha : F(a, b) \rightarrow \mathbb{Z}^2$  be the abelianization homomorphism, that is,  $\alpha(a) = (1, 0)$  and  $\alpha(b) = (0, 1)$ . Then  $\alpha$  is constant on every conjugacy class and therefore  $\alpha$  defines a map  $\beta : \mathcal{C}_2 \rightarrow \mathbb{Z}^2$ .

**Definition 3.2** (Visible points). A point  $(p, q) \in \mathbb{Z}^2$  is called *visible* if  $\gcd(p, q) = 1$ . We denote the set of all visible points in  $\mathbb{Z}^2$  by  $V$ .

We will need the following known facts about primitive elements in  $F(a, b)$  (see, for example, [5, 12]):

**Proposition 3.3.** *The following hold:*

- (1) *The restriction of  $\beta$  to  $\mathcal{P}_2$  is a bijection between  $\mathcal{P}_2$  and the set of visible elements  $V \subseteq \mathbb{Z}^2$ .*
- (2) *Let  $w \in F(a, b)$  be a cyclically reduced primitive element and let  $\alpha(w) = (p, q) \in \mathbb{Z}^2$ .*

*Then every occurrence of  $a$  in  $w$  has the same sign (either  $-1, 0$  or  $1$ ) as  $p$  and every occurrence of  $b$  in  $w$  has the same sign (again either  $-1, 0$  or  $1$ ) as  $q$ . Thus the total number of occurrences of  $a^{\pm 1}$  in  $w$  is equal to  $|p|$  and the total number of occurrences of  $b^{\pm 1}$  in  $w$  is equal to  $|q|$ .*

**Definition 3.4** (Admissible function). We say that a function  $f : (0, \infty) \rightarrow [0, \infty)$  is *admissible* if it satisfies the following conditions:

- (1) We have  $f''(x) > 0$  for every  $x > 0$ .
- (2) There is some  $\epsilon > 0$  such that  $\lim_{x \rightarrow \infty} x^{3+\epsilon} f(x) = 0$ .

The second condition means that  $f(x)$  converges to zero asymptotically faster than  $\frac{1}{x^{3+\epsilon}}$  as  $x \rightarrow \infty$ . Note that an admissible function must be strictly positive and monotone decreasing on  $(0, \infty)$ .

**Theorem 3.5.** *Let  $f$  be any admissible function. Then the following hold:*

- (1) *We have  $0 < P_f(t) < \infty$  for every  $t \in (0, 1)$ .*
- (2) *The function  $P_f(t)$  is strictly convex on  $(0, 1)$  and achieves a unique minimum at  $t = 1/2$ . In particular,  $P_f(t)$  is not a constant locally near  $t = 1/2$ .*

*Proof.* For every  $(p, q) \in V$  and  $t \in (0, 1)$  denote

$$g_{p,q}(t) = f(t|p| + (1-t)|q|) + f(t|q| + (1-t)|p|).$$

Note that if  $(p, q) \in V$  then  $\gcd(p, q) = 1$  and hence  $|p| \neq |q|$ . We can therefore partition  $V$  as the collection of pairs  $(p, q), (q, p)$  of visible elements and every such pair has a unique representative where the absolute value of the first coordinate is bigger than that of the second coordinate.

Let  $w \in \mathcal{P}_2$  be arbitrary and let  $(p, q) = \beta(w)$ . Proposition 3.3 and the definition of  $\mathcal{L}_t$  imply that for any  $t \in (0, 1)$  we have

$$\ell_{\mathcal{L}_t}(w) = t|p| + (1-t)|q|.$$

Let  $V' := \{(p, q) \in V : |p| > |q|\}$ . Then we have

$$\begin{aligned} (\dagger) \quad P_f(t) &= \sum_{w \in \mathcal{P}_2} f(\ell_{\mathcal{L}_t}(w)) = \sum_{(p,q) \in V} f(t|p| + (1-t)|q|) = \\ &= \sum_{(p,q) \in V'} f(t|p| + (1-t)|q|) + f(t|q| + (1-t)|p|) = \sum_{(p,q) \in V'} g_{p,q}(t). \end{aligned}$$

Fix some  $t \in (0, 1)$ . We can also represent  $P_f(t)$  as

$$P_f(t) = \sum_{N=1}^{\infty} \sum_{(p,q) \in V, \max\{|p|, |q|\} = N} f(t|p| + (1-t)|q|).$$

Since  $f(x)$  is a monotone non-increasing function, if  $(p, q) \in V, \max\{|p|, |q|\} = N$ , we have

$$f(t|p| + (1-t)|q|) \leq \min\{f(tN), f((1-t)N)\} = f(cN)$$

where  $c = \max\{t, 1-t\}$ . For every integer  $N \geq 1$  the number of points  $(p, q) \in \mathbb{Z}^2$  with  $|p| \leq N, |q| \leq N$  is  $(2N+1)^2$ .

Therefore

$$\begin{aligned} P_f(t) &= \sum_{N=1}^{\infty} \sum_{(p,q) \in V, \max\{|p|, |q|\} = N} f(t|p| + (1-t)|q|) \leq \\ &= \sum_{N=1}^{\infty} \sum_{(p,q) \in V, \max\{|p|, |q|\} = N} f(cN) \leq \sum_{N=1}^{\infty} (2N+1)^2 f(cN) < \infty \end{aligned}$$

because of condition (2) in the definition of admissibility of  $f(x)$ . Thus  $0 < P_f(t) < \infty$  for every  $t \in (0, 1)$ .

Note that for each  $(p, q) \in V'$

$$\begin{aligned} g'_{p,q}(t) &= f'(t|p| + (1-t)|q|)(|p| - |q|) + f'(t|q| + (1-t)|p|)(|q| - |p|) \\ g''_{p,q}(t) &= f''(t|p| + (1-t)|q|)(|p| - |q|)^2 + f''(t|q| + (1-t)|p|)(|q| - |p|)^2 \end{aligned}$$

Since  $|p| > |q|$  and, by definition of admissibility,  $f''(x) > 0$  for every  $x \in \mathbb{R}$ , we conclude that  $g''_{p,q}(t) > 0$  for every  $t \in (0, 1)$ . Hence  $g_{p,q}$  is strictly convex on  $(0, 1)$ . Moreover,

$$g'_{p,q}\left(\frac{1}{2}\right) = f'\left(\frac{|p|}{2} + \frac{|q|}{2}\right)(|p| - |q|) + f'\left(\frac{|q|}{2} + \frac{|p|}{2}\right)(|q| - |p|) = 0.$$

Since  $g''_{p,q} > 0$  on  $(0, 1)$ , it follows that  $g_{p,q}$  is strictly convex on  $(0, 1)$  and achieves a unique minimum on  $(0, 1)$  at  $t = \frac{1}{2}$ .

Since  $0 < P_f < \infty$  on  $(0, 1)$  and  $P_f = \sum_{(p,q) \in V'} g_{p,q}$ , it also follows that  $P_f$  is strictly convex on  $(0, 1)$  and achieves a unique minimum on  $(0, 1)$  at  $t = \frac{1}{2}$ .  $\square$

#### 4. EXPLOITING CONVEXITY

In this section we combine the ideas of the previous two sections and establish Theorem C from the introduction.

**Theorem 4.1.** *Let  $k \geq 2$  and let  $f : (0, \infty) \rightarrow (0, \infty)$  be monotone decreasing function such that the following hold:*

- (1) *The function  $f(x)$  is strictly convex on  $(0, \infty)$ .*
- (2)

$$\limsup_{x \rightarrow \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$

*Then there exists a convex neighborhood  $U$  of  $\mathcal{L}_*$  in  $\Delta_k$  such that  $0 < P_f < C_f < \infty$  on  $U$  and both  $C_f$  and  $P_f$  are strictly convex on  $U$ . In particular,  $C_f \neq \text{const}$  on  $U$  and  $P_f \neq \text{const}$  on  $U$ .*

*Proof.* By Theorem 2.4 there exists a convex neighborhood  $U$  of  $\mathcal{L}_*$  in  $\Delta_k$ , such that  $0 < P_f < C_f < \infty$  on  $U$ . We will prove that  $P_f$  and  $C_f$  are strictly convex on  $U$ .

Let  $D$  be the set of all  $k$ -tuples of integers  $m = (m_1, \dots, m_k)$  such that  $m_i \geq 0$  for  $i = 1, \dots, k$  and  $m_1 + \dots + m_k > 0$ . For each  $m = (m_1, \dots, m_k) \in D$  let  $Q_m$  be the set of all  $w \in \mathcal{C}_k$  such that  $w$  involves exactly  $m_i$  occurrences of  $a_i^{\pm 1}$  for  $i = 1, \dots, k$  and let  $q_m := \#(Q_m)$ . Note that for every  $w \in Q_m$ , if  $\mathcal{L} = (x_1, \dots, x_k) \in \Delta_k$ , then we have

$$\ell_{\mathcal{L}}(w) = m_1 x_1 + \dots + m_k x_k.$$

Denote by  $f_m : \Delta_k \rightarrow \mathbb{R}$  the function defined as

$$f_m(x_1, \dots, x_k) := f(m_1 x_1 + \dots + m_k x_k), \quad (x_1, \dots, x_k) \in \Delta_k.$$

The function  $f(x)$  is convex on  $(0, \infty)$  and the function  $(x_1, \dots, x_k) \mapsto m_1 x_1 + \dots + m_k x_k$  is linear on  $\Delta_k$ . Therefore  $f_m$  is convex on  $\Delta_k$ .

Then for any  $\mathcal{L} = (x_1, \dots, x_k) \in \Delta_k$  we have

$$C_f(\mathcal{L}) = \sum_{m \in D} q_m f_m(\mathcal{L}).$$

Since each  $f_m$  is convex on  $\Delta_k$ , it follows that  $C_f$  is convex on  $\Delta$ . We claim that  $C_f$  is strictly convex on  $U$ . Let  $D_1$  be the subset of  $D$  consisting of all the  $k$ -tuples having a single nonzero entry equal to 1, that is,  $D_1$  is the union of the  $k$  standard unit vectors in  $\mathbb{Z}^k$ . Let  $m_i = (0, \dots, 1, \dots, 0) \in D_1$  where 1 occurs in the  $i$ -th position. Then  $Q_{m_i} = \{[a_i], [a_i^{-1}]\}$  and  $q_{m_i} = 2$ . Also,  $f_{m_i}(x_1, \dots, x_k) = f(x_i)$  for every  $(x_1, \dots, x_k) \in \Delta_k$ .

Put  $g := f_{m_1} + \dots + f_{m_k} : \Delta_k \rightarrow \mathbb{R}$ , so that

$$g(x_1, \dots, x_k) = f(x_1) + \dots + f(x_k), \quad (x_1, \dots, x_k) \in \Delta_k.$$

It is easy to see that  $g$  is strictly convex on  $\Delta_k$  since  $f$  is strictly convex on  $(0, \infty)$ . We have:

$$C_f = \sum_{m \in D} q_m f_m = 2g + \sum_{m \in D - D_1} q_m f_m$$

Since  $C_f < \infty$  on a convex set  $U$  and since  $g$  is strictly convex on  $U$  and  $\sum_{m \in D-D_1} q_m f_m$  is convex on  $U$ , it follows that  $C_f$  is strictly convex on  $U$  as claimed.

The proof that  $P_f$  is strictly convex on  $U$  is exactly the same as for  $C_f$  above. The only change that needs to be made is to re-define  $Q_m$  for each  $m = (m_1, \dots, m_k) \in D$  as the set of all  $w \in \mathcal{P}_k$  such that  $w$  involves exactly  $m_i$  occurrences of  $a_i^{\pm 1}$  for  $i = 1, \dots, k$ .  $\square$

**Remark 4.2.** Let  $\mathcal{Z}_k$  be the set of all *root-free* conjugacy classes  $w \in \mathcal{C}_k$ , that is, conjugacy classes of nontrivial elements of  $F_k$  that are not proper powers. It is not hard to show, similar to Proposition 2.3, that if  $\mathcal{L}$  is a metric structure on  $W_k$  then

$$h_{\mathcal{L}} = \tilde{h}_{\mathcal{L}}$$

where

$$\tilde{h}_{\mathcal{L}} := \lim_{R \rightarrow \infty} \frac{\log \#\{w \in \mathcal{Z}_k : \ell_{\mathcal{L}}(w) \leq R\}}{R}.$$

If one now re-defines the McShane function  $C_f$  as  $S_f$ :

$$S_f : \Delta_k \rightarrow (0, \infty], \quad S_f(\mathcal{L}) = \sum_{w \in \mathcal{Z}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k,$$

then the proofs of the parts of Theorem A and Theorem C dealing with  $C_f$  go through verbatim for  $S_f$ .

#### REFERENCES

- [1] H. Akiyoshi, H. Miyachi, and M. Sakuma, *Variations of McShane's identity for punctured surface groups*, London Mathematical Society Lecture Notes, Y. Minsky, M. Sakuma and C. Series (Eds.), Cambridge University Press (to appear).
- [2] H. Akiyoshi, H. Miyachi, and M. Sakuma, *A refinement of McShane's identity for quasi-fuchsian punctured torus groups*. In the tradition of Ahlfors and Bers, III, 21–40, *Contemp. Math.*, 355, Amer. Math. Soc., Providence, RI, 2004
- [3] B. H. Bowditch, *A proof of McShane's identity via Markoff triples*. *Bull. London Math. Soc.* **28** (1996), no. 1, 73–78
- [4] B. H. Bowditch, *A variation of McShane's identity for once-punctured torus bundles*. *Topology* **36** (1997), no. 2, 325–334
- [5] M. Cohen, W. Metzler, and A. Zimmermann, *What does a basis of  $F(a, b)$  look like?* *Math. Ann.* **257** (1981), no. 4, 435–445
- [6] I. Kapovich and T. Nagnibeda, *The Patterson-Sullivan embedding and minimal volume entropy for outer space*, preprint; <http://arxiv.org/abs/math.GR/0504445>
- [7] I. Kapovich, P. Schupp and V. Shpilrain, *Generic properties of Whitehead's Algorithm and isomorphism rigidity of random one-relator groups*, *Pacific J. Math.*, to appear
- [8] S. Lim, *Minimal volume entropy on graphs*, preprint, 2005, <http://www.arxiv.org/abs/math.GR/0506216>
- [9] G. McShane, *A remarkable identity for lengths of curves*, Thesis, Univ. Warwick, Coventry, 1991
- [10] G. McShane, *Simple geodesics and a series constant over Teichmuller space*. *Invent. Math.* **132** (1998), no. 3, 607–632
- [11] I. Rivin, *Growth on groups (and other stories)*, preprint, 1999, <http://www.arxiv.org/abs/math.CO/9911076>
- [12] R. Osborne, and H. Zieschang, *Primitives in the free group on two generators*. *Invent. Math.* **63** (1981), no. 1, 17–24
- [13] Ser Peow Tan, Yan Loi Wong, Ying Zhang, *Generalizations of McShane's identity to hyperbolic cone-surfaces*, preprint, 2004, <http://arxiv.org/abs/math.GT/0404226>

- [14] Ser Peow Tan, Yan Loi Wong, Ying Zhang, *McShane's identity for classical Schottky Groups*, preprint, 2004,  
<http://arxiv.org/abs/math.GT/0411628>

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